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INTEGRABLE SYSTEMS AND ALGEBRAIC SURFACES

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1. Introduction. Of the completely integrable Hamiltonian systems, those which are algebraically integrable [AvM1], [AvM2], [vM] occupy a privileged position. Roughly speaking, these are complex integrable systems, such that the joint level sets of the Hamiltonians when compactified and desingularized are Abelian varieties, in such a way that the linear structure given by the Hamiltonian flow is that of the Abelian variety. These systems have various desirable properties, such as the Painlevé property [AvM1], and comprise many of the most interesting classical examples.

A particularly nice class of examples is given by some natural systems on the coadjoint orbits of a loop algebra $\tilde{\mathfrak{g}}_+$ of polynomials in one variable λ with values in a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ [AvM3], [AHP], [B], [FRS], [RS]. (We will return to these systems at greater length in Section 4, and will simply summarize here some of their features, as these systems are our motivating examples.) Via a trace residue pairing, the dual of $\tilde{\mathfrak{g}}_+$ can be identified with the Lie algebra $\tilde{\mathfrak{g}}_-$ of power series in λ^{-1}. One can show [H] that the finite-dimensional orbits in $\tilde{\mathfrak{g}}_- \simeq (\tilde{\mathfrak{g}}_+)^*$ are of elements of the form

$$N(\lambda) = \frac{L(\lambda)}{a(\lambda)} \tag{1.1}$$

where $a(\lambda) = \prod_{i=1}^{n} (\lambda - \alpha_i) \in \mathbb{C}[\lambda]$, $L(\lambda) \in \widetilde{\mathfrak{sl}}(r)_+$, degree $(L(\lambda)) <$ degree $(a(\lambda))$, and one expands around $\lambda = \infty$ to obtain a series in λ^{-1}. We note that the group G corresponding to $\mathfrak{g}$ acts naturally on these orbits, and one can reduce the orbits by this action.

Let us take $\mathfrak{g}$ to be $\mathfrak{sl}(r, \mathbb{C})$. At the heart of the integrable systems one considers on the orbit are the spectral curves $\mathscr{S}_0$ in $\mathbb{C}^2$, cut out by

$$\det(N(\lambda) - z\mathbb{I}) = 0. \tag{1.2}$$

The commuting Hamiltonians considered are the coefficients of the polynomials defining the spectral curve. They are invariant under the G-action.

To $N(\lambda)$, one can also associate a line bundle E, given over S_0 as the cokernel of $(N(\lambda) - z\mathbb{I})$. This bundle extends naturally to a natural compactification S of S_0. [AHH1] On the reductions $\mathscr{R}$ of the orbits by the $SL(r, \mathbb{C})$ action, the pair

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