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STRUCTURE OF TILINGS OF THE LINE BY A FUNCTION

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1. Introduction. Traditional tiling problems concern whether a subset S of $\mathbb{R}^n$ can be tiled using a given set of allowed tile shapes ("prototiles"). Such problems may be reformulated as expressing the characteristic function χ_S as a sum of characteristic functions of sets isometric to prototiles under the allowed group of tile motions. A natural generalization is to relax this condition to allow tilings of χ_S using more general functions. The supports of the copies of the functions used in such a tiling may overlap ("soft tiles"). Soft tilings are a special case of "soft packings," which have been studied to obtain bounds in sphere packing and coding theory [9]. Translation tilings by functions arise naturally in wavelet theory: the scaling function for a compactly supported wavelet basis of $\mathbb{R}^n$ given by a multiresolution analysis must always have a lattice tiling of $\mathbb{R}^n$ in this generalized sense, see Strichartz [13, 1.17]. Such tilings also arise in subdivision schemes in curve and surface design and in approximation, see [2, p. 14]. Multiple tilings using copies of a set T are a special case of tilings by functions, in which the functions used are scaled characteristic functions $M^{-1}\chi_T$, where M is the multiplicity.

This paper studies tilings of the line $\mathbb{R}$ by translates of a single function $f \in L^1(\mathbb{R})$. A tile set A gives a *general tiling of (constant) weight* w provided that

$$\sum_{a \in A} f(x + a) = w, \tag{1.1}$$

for almost every (Lebesgue) $x \in \mathbb{R}$, where the convergence in (1.1) is absolute. The tile set A is required to be discrete; i.e., for each $T > 0$ the set $\{a \in A: |a| < T\}$ is finite, and we allow elements of A to occur with finite multiplicity. Our object is to determine which functions tile $\mathbb{R}$ and to specify the structure of possible tilings.

General tilings by a function f include the possibility that mass can "leak out to infinity"; cf. example 7.1 in §7. To exclude this pathology we restrict the class of allowed tilings. A tile set A is of *bounded density* if there is a constant $C > 0$ such that for all $T \in \mathbb{R}$,

$$\#\{a \in A: T \leq a < T + 1\} \leq C,$$

and the associated tiling (1.1) is called a *tiling of bounded density.* For nonnegative functions any general tiling is of bounded density; see Lemma 2.1.

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