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CANONICAL BASES AND SELF-EVACUATING TABLEAUX

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0. Introduction. The main purpose of this paper is to present some further examples of the "$q = -1$ phenomenon" introduced in [Ste]. The examples we present here rely partly on some recent work of Berenstein and Zelevinsky [BZ] that relates the evacuation operation on semistandard tableaux to the action of a fundamental involution on canonical bases for irreducible $U_q(sl_n)$-modules, as developed by Lusztig [L].

By the "$q = -1$ phenomenon" we mean a situation of the following type. One has a set of combinatorial objects B (such as tableaux), together with a generating function $F(q)$ that enumerates the objects in B according to some weight depending on q. (The use of q here is not intended to suggest any relationship to the quantum parameter q.) The $q = -1$ phenomenon occurs when there is a "natural" involution on B such that $F(-1)$ is the number of fixed points of the involution. If there is a closed formula for $F(q)$, then there is hence also a closed formula for the number of fixed points.

In this paper, the set B is a canonical basis for an irreducible $\mathfrak{g}$-module V, where $\mathfrak{g}$ is a (complex, semisimple) Lie algebra. The action of $\mathfrak{g}$ of course lifts naturally to the associated simply connected Lie group $\tilde{G}$. In this situation, there exist elements $x = x(q)$ in the Cartan subgroup of $\tilde{G}$ with the property that the trace of $x(q)$ on V can be computed via the q-analogue of the Weyl dimension formula. On the other hand, since the Cartan subgroup acts diagonally with respect to the basis B, the trace of $x(q)$ can be viewed as a generating function $F(q)$ for B. Building on previous work in [Ste], we prove (Theorem 1.2) that there is an element of $\tilde{G}$ that is conjugate to $x(-1)$ in $\tilde{G}$ and whose action on V, up to a scalar of unit absolute value, agrees with a certain fundamental involution on B studied by Lusztig [L, §3]. Thus, up to a scalar factor, $F(-1)$ is the number of fixed points of Lusztig's involution, and the net result is that we have obtained a large class of examples of the $q = -1$ phenomenon.

In Section 2, we analyze in detail the number of fixed points of Lusztig's involution. We characterize (Theorem 2.1) when it is nonzero, and show that in general (Theorem 2.2), it can be expressed in terms of dimensions of irreducible representations of a Lie algebra $\mathfrak{g}_{(2)}$ whose root system is contained in the root system of $\mathfrak{g}$. In the simply laced case, $\mathfrak{g}_{(2)}$ can be naturally realized as a Lie subalgebra of $\mathfrak{g}$, but not in general (see Remark 2.3(b)). In the case $\mathfrak{g} = sl(n)$, the sit-

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