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CONVERGENCE OF ZETA FUNCTIONS ON SYMPLECTIC AND METAPLECTIC GROUPS

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Introduction. Each of our zeta functions is associated with a holomorphic Hecke eigenform f of integral or half-integral weight with respect to a congruence subgroup of $G^n = Sp(n, F)$, where F is a totally real algebraic number field. The form f can be considered on $G^n_{\mathbf{A}}$ or on the metaplectic cover $M^n_{\mathbf{A}}$ of $G^n_{\mathbf{A}}$ accordingly. The zeta function has the Euler product expression

$$Z(s) = \prod_{\mathfrak{p}} W_{\mathfrak{p}}(N(\mathfrak{p})^{-s})^{-1}, \tag{1}$$

where $\mathfrak{p}$ runs over all the prime ideals of F, and $W_{\mathfrak{p}}$, except finitely many $\mathfrak{p}$'s, is a polynomial of degree $2n + 1$ or $2n$ according as the weight is integral or half-integral. It may be noted that such Euler products on $M^n_{\mathbf{A}}$ and their meromorphic continuation have been obtained in our recent paper [S10]. Those on $G^n_{\mathbf{A}}$ are well known (cf. the introduction of [S7]).

Now our first main purpose is to show that the right-hand side of (1) is absolutely convergent, and consequently $Z(s) \neq 0$, for $\mathrm{Re}(s) > (3n/2) + 1$ (Theorem A). Here, for some technical reasons, we take $s = n + 1/2$ to be the center of the critical strip. Duke, Howe, and Li showed in [DHL] that if the form is on $Sp(n, \mathbf{Q})_{\mathbf{A}}$, then the absolute convergence holds for $\mathrm{Re}(s) > (5n/2) + 1$ in general, and in particular for $\mathrm{Re}(s) > (3n/2) + 1$ if $n = 2^r$ with $0 < r \in \mathbf{Z}$. Our present result applies to every n, and even to the Euler products on $M^n_{\mathbf{A}}$.

The bound $(3n/2) + 1$ is best possible, since the right-hand side of (1) does not converge at this point for a certain f. This fact was shown in [DHL] for even n as a consequence of a result of Rallis. We shall prove more generally that given any n, Z has a pole at $(3n/2) + 1$ only if the weight of f is of a "relatively small" restricted type, and it must be integral or half-integral according as n is even or odd, and moreover that such a pole occurs for every n with a certain theta series as f (Theorem C). In [S8] and [S10], we obtained some related results on the location of possible poles of Z. We shall state the results in more refined forms as Theorems B1 and B2. In this and other problems in the present paper, we consider not only Z itself but also its twists by Hecke characters of F.

As an application of Theorem A, we shall show that if the weight is "not too small," the space $\mathscr{M}^n_k(\Gamma)$ of all holomorphic modular forms of weight k with respect to a congruence subgroup Γ of G^n is spanned by cusp forms and Eisenstein

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