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HOLOMORPHIC SYMPLECTOMORPHISMS IN $\mathbb{C}^{2p}$

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1. Introduction. We will discuss biholomorphic symplectomorphisms of $\mathbb{C}^{2p}$ and study the abundance or nonabundance of periodic or quasi-periodic orbits.

Let $\Omega := \sum dz_j \wedge dw_j$ where (z, w), $z = (z_1, \ldots, z_p)$, $w = (w_1, \ldots, w_p)$ denotes coordinates in $\mathbb{C}^{2p}$. A biholomorphic map $f\colon \mathbb{C}^{2p} \mapsto \mathbb{C}^{2p}$ is a symplectomorphism if and only if $f^*(\Omega) = \Omega$. In the second paragraph, we make some preliminary remarks on biholomorphic symplectomorphisms of $\mathbb{C}^{2p}$.

In the third paragraph, we study the orbits of a time-independent, holomorphic Hamiltonian in $\mathbb{C}^{2p}$. We show that generically all orbits go to ∞; see Theorem 3.4.

Paragraph 4 is devoted to a question of Herman. Let S be the space of holomorphic symplectomorphisms of $\mathbb{C}^{2p}$ with the topology of uniform convergence on compact sets. For $f \in S$, let $K_f := \{(z, w); f^n(z, w) \text{ is bounded}\}$.

CONJECTURE 1.1 (Herman). *There is a G_δ dense set $S' \subset S$ such that for $f \in S'$, K_f has empty interior.*

We confirm Herman's conjecture. The case $p = 1$ was done previously in [FS]. The answer to Herman's question in $\mathbb{R}^{2p}$, $p > 1$, is not known.

2. Holomorphic maps as Hamiltonian flows. Let E denote the space of entire holomorphic functions. We consider each $F \in E$ as a holomorphic Hamiltonian giving rise to a holomorphic vector field $X = X_F = (-\partial F/\partial w_1, \ldots, -\partial F/\partial w_p, \partial F/\partial z_1, \ldots, \partial F/\partial z_p)$.

Let S denote the space of biholomorphic symplectomorphisms of $\mathbb{C}^{2p}$. We give S the topology of uniform convergence on compact sets. The following proposition is standard.

PROPOSITION 2.1. *Let $\Phi \in S$. Then there exists a C^∞ function $F(z, w, t)\colon \mathbb{C}^{2p} * \mathbb{R} \mapsto \mathbb{C}$ with period 1 in t such that $F_t(z, w) := F(z, w, t)$ is holomorphic for each t. Moreover, Φ is the time 1 map of the time-dependent holomorphic Hamiltonian vector field*

$$X_t = (-\partial F/\partial w_1, \ldots, -\partial F/\partial w_p, \partial F/\partial z_1, \ldots, \partial F/\partial z_p).$$

Proof. First, we pick a C^∞ map $G\colon [0, 1] \mapsto S$ so that $G(t) \equiv \mathrm{Id}$ for t small and $G(t) \equiv \Phi$ for t close to 1. We can write

$$G(t)(z, w) = (P_1(z, w, t), \ldots, P_p(z, w, t), Q_1(z, w, t), \ldots, Q_p(z, w, t)).$$

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