Vol. 82, No. 1 DUKE MATHEMATICAL JOURNAL © January 1996

MULTIPLICITIES FORMULA FOR GEOMETRIC QUANTIZATION, PART I

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1. Introduction. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ acting on a compact symplectic manifold M by a Hamiltonian action. If $X \in \mathfrak{g}$, we denote by X_M the vector field on M induced by the action of G. We denote by σ the symplectic form on M and by $\mu: M \to \mathfrak{g}^*$ the moment map. To simplify, we will assume in this article that M has a G-invariant spin structure. We will show in the appendix how to remove this assumption.

Let us assume that M is prequantized, and let $\mathscr{L}$ be the Kostant-Souriau line bundle on M. We denote by $R(G)$ the ring of virtual finite-dimensional representations of G. An element of $R(G)$ is thus a difference of two finite-dimensional representations of G. We associate to $(M, \mathscr{L})$ a virtual representation $Q(M, \mathscr{L}) \in R(G)$ of G constructed as follows: Choose a G-invariant Riemannian structure on M. Let $\mathscr{S}^{\pm}$ be the half-spin bundles over M determined by the spin structure and the symplectic orientation of M. Let $\Gamma(M, \mathscr{S}^{\pm} \otimes \mathscr{L})$ be the spaces of smooth sections of $\mathscr{S}^{\pm} \otimes \mathscr{L}$. Consider the twisted Dirac operator

$$D_{\mathscr{L}}^{+}: \Gamma(M, \mathscr{S}^{+} \otimes \mathscr{L}) \to \Gamma(M, \mathscr{S}^{-} \otimes \mathscr{L}).$$

This is an elliptic operator commuting with the action of G. We define a virtual representation $Q(M, \mathscr{L})$ of G by the formula:

$$Q(M, \mathscr{L}) = (-1)^{\dim M/2}([\operatorname{Ker} D_{\mathscr{L}}^{+}] - [\operatorname{Coker} D_{\mathscr{L}}^{+}]).$$

The virtual representation $Q(M, \mathscr{L})$ so obtained is independent of the choice of the Riemannian structure on M. If M and $\mathscr{L}$ have G-invariant complex structure, then $Q(M, \mathscr{L})$ (apart from a shift of parameters) is the direct image of the sheaf $\mathcal{O}(\mathscr{L})$ of holomorphic sections of $\mathscr{L}$ by the map $M \to$ *point*. In the differentiable category, we employ as in Atiyah-Hirzebruch [3] the Dirac operator to define the direct image $Q(M, \mathscr{L}) \in R(G) = K_G(\textit{point})$ of $\mathscr{L} \in K_G(M)$. If the group G is trivial, then $Q(M, \mathscr{L}) \in \mathbb{Z}$ is the index of the operator $D_{\mathscr{L}}^{+}$. We call this number the Riemann-Roch number of $(M, \mathscr{L})$.

We are interested in describing the decomposition of $Q(M, \mathscr{L})$ in irreducible representations of G. Let $G = T$ be a torus. Let $P \subset i\mathfrak{t}^*$ be the lattice of weights of T. We have a decomposition

$$Q(M, \mathscr{L}) = \sum_{\xi \in iP} n(\xi, M, \mathscr{L}) e_{i\xi},$$

Received 17 October 1994. Revision received 16 May 1995.