INEQUALITIES FOR SECOND-ORDER ELLIPTIC EQUATIONS WITH APPLICATIONS TO UNBOUNDED DOMAINS I

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1. Introduction. In recent papers [BCN2], [BCN3], the authors have studied symmetry and monotonicity properties for positive solutions u of elliptic equations of the form

$$(1.1) \qquad \begin{aligned} u > 0, \quad \Delta u + f(u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

in several classes of unbounded domains Ω in $\mathbb{R}^n$. These papers extended some of the results for bounded domains of [GNN] and [BN2].

For example, in [BCN2] we considered a half space, $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_n > 0\}$. Under assumptions on f, we showed that a bounded solution of (1.1) is necessarily a function of x_n alone (symmetry) and is increasing in x_n. In [BCN3], we considered Ω bounded by a Lipschitz graph,

$$\Omega = \{x \in \mathbb{R}^n; x_n > \varphi(x_1, \ldots, x_{n-1})\}, \qquad \varphi\colon \mathbb{R}^{n-1} \to R \quad \text{Lipschitz}$$

and proved monotonicity for any bounded solution of (1.1), namely, that $u_{x_n} > 0$ in Ω. Proofs relied on the moving plane method and the sliding method.

Here we continue this program by considering another type of bounded domain,

$$\Omega = \mathbb{R}^{n-j} \times \omega$$

where ω is a smooth bounded domain in $\mathbb{R}^j$. We denote by $x = (x_1, \ldots, x_{n-j})$ the coordinates in $\mathbb{R}^{n-j}$, and by $y = (y_1, \ldots, y_j)$ the coordinates in ω.

Our goal is to establish symmetry of solutions of (1.1) corresponding to symmetries of ω. For example, if ω is a ball $\{|y| < R\}$, we prove that any solution of (1.1) depends only on $|y|$ and x, and is decreasing in $|y|$. Note that u is not assumed to be bounded. Throughout the paper we assume that f is Lipschitz continuous, with Lipschitz constant k, on R^+ (or on $[0, \sup u]$ in the case where u

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