ZEROS OF PRINCIPAL L-FUNCTIONS AND RANDOM MATRIX THEORY

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1. Introduction. Our goal in this paper is to study the distribution of zeros of the Riemann zeta function as well as of more general L-functions.[1] According to conjectures of Langlands [14], the most general L-function is that attached to an automorphic representation of GL_N over a number field, and these in turn should be expressible as products of the "standard" L-functions $L(s, \pi)$ attached to *cuspidal* automorphic representations of GL_m over the *rationals*. Such L-functions are therefore believed to be the building blocks for general L-functions, and we call them (principal) *primitive L-functions* of degree m. (They do not factor as products of such L-functions.)[2] For $m = 1$ these are the Riemann zeta function $\zeta(s)$ and Dirichlet L-functions $L(s, \chi)$ with χ primitive. For $m = 2$ the analytic properties and functional equation of such L-functions were investigated by Hecke and Maass, and for $m \geqslant 3$ by Godement and Jacquet [5]. We are interested in the fine structure of the distribution of the nontrivial zeros of such primitive $L(s, \pi)$. Let $\rho^{(\pi)} = (1/2) + i\gamma^{(\pi)}$ denote these zeros. To motivate the formulation of our results, we begin by assuming the Riemann hypothesis (RH) for $L(s, \pi)$, that is, that $\gamma^{(\pi)} \in \mathbf{R}$. We order the $\gamma^{(\pi)}$'s (with multiplicities)

$$\cdots \leqslant \gamma_{-2}^{(\pi)} \leqslant \gamma_{-1}^{(\pi)} < 0 \leqslant \gamma_1^{(\pi)} \leqslant \gamma_2^{(\pi)} \cdots .$$

The number of γ's in an interval $[T, T + 1]$ is asymptotic to $(m/2\pi) \log T$ as $T \to \infty$ (see (2.11)). It follows that the numbers $\tilde{\gamma}_j^{(\pi)} = (m/2\pi)\gamma_j \log|\gamma_j|$ have unit mean spacing. The problem is to understand the statistical nature of the sequence $\tilde{\gamma}_j^{(\pi)}$: Do they come down randomly (Poisson process) or do they follow a more revealing distribution?

In the case of the Riemann zeta function, following the original calculation by Montgomery [20] of the pair correlation (see below) and the extensive numerical calculations of Odlyzko [21], [22], it is now well accepted (but far from proven) that the consecutive spacings follow the Gaussian unitary ensemble (GUE) distribution from random matrix theory. That is, if $\delta_n = \tilde{\gamma}_{n+1} - \tilde{\gamma}_n$ are the normalized

[1] The reader interested only in the Riemann zeta function $\zeta(s)$ should read the paper with $L(s, \pi)$ replaced by $\zeta(s)$ and $m = 1$ everywhere, in which case the results were announced in [26].

[2] It is quite plausible that these coincide with the primitive Dirichlet series introduced by Selberg [29] or the "arithmetic Dirichlet series" in Piatetski-Shapiro [24].

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