BERNSTEIN'S INEQUALITY AND THE RESOLUTION OF SPACES OF ANALYTIC FUNCTIONS

CHARLES FEFFERMAN AND RAGHAVAN NARASIMHAN

0. Introduction. The growth and smoothness properties of polynomials F on $\mathbf{R}^n$ are controlled by standard inequalities of the following type:

$$\sup_{B(x,\rho)} |\nabla F| \leqslant \frac{C}{\rho} \sup_{B(x,\rho)} |F| \tag{0.1}$$

$$\sup_{B(x,\rho)} |F| \leqslant C \sup_{B(x,\rho/2)} |F| \tag{0.2}$$

$$\sup_{B(x,\rho)} |F| \leqslant \frac{C}{\rho^n} \int_{B(x,\rho)} |F(y)|\, dy \tag{0.3}$$

$$\sup_{B_C(x,\rho)} |F| \leqslant C \sup_{B(x,\rho)} |F|. \tag{0.4}$$

Here, C denotes a constant depending only on the degree of F, while $B(x, \rho)$, $B_c(x, \rho)$ denote the ball, with center x and radius ρ in $\mathbf{R}^n$ and $\mathbf{C}^n$, respectively.

We call inequalities (0.1)–(0.4) the *Bernstein inequalities.*

In this article, we shall show that if V_λ is a finite-dimensional vector space of real analytic functions of n variables depending real-analytically on a parameter $\lambda \in \mathbf{R}^m$, then the Bernstein inequalities (0.1)–(0.4) continue to hold for $F \in V_\lambda$, *locally uniformly with respect to* λ. Thus, our first main result is as follows.

BERNSTEIN THEOREM. *Let $F_{1,\lambda}, \dots, F_{N,\lambda}$ be holomorphic functions on the complex ball $B_c(0, 1 + \varepsilon)$, $\varepsilon > 0$, in $\mathbf{C}^n$ depending real-analytically on $\lambda \in U \subset \mathbf{R}^m$ (U an open set). Let V_λ be the linear span of the $F_{k,\lambda}$, $1 \leqslant k \leqslant N$.*

Then, for any compact set $K \subset U$, there is a constant $C > 0$ such that the Bernstein inequalities (0.1)–(0.4) hold for any $F \in V_\lambda$, $\lambda \in K$, and $B(x, \rho) \subset B(0, 1)$.

For example, if $\lambda = (\xi_1, \dots, \xi_N) \in (\mathbf{R}^n)^N$ and $V_\lambda = \operatorname{span}\{e^{\langle \xi_1, x\rangle}, \dots, e^{\langle \xi_N, x\rangle}\}$, the above theorem asserts that (1)–(4) hold for $F \in V_\lambda$ with a constant C depending only on upper bounds for $|x|$, ρ, $|\xi_1|, \dots, |\xi_N|$.

The difficulty in proving these uniform estimates comes from the fact that V_λ may degenerate (i.e., that its dimension may drop). This difficulty arises already if we attempt to prove (0.1)–(0.4) for a single vector space V_{λ_0} because the estimates

Received 5 August 1994.
Fefferman supported in part by a grant from the National Science Foundation.