# A SUFFICIENT CONDITION FOR INVARIANCE OF ESSENTIAL COMPONENTS

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**Introduction.** Key advances in the study of Nash equilibria of finite games are the articles by Kohlberg and Mertens [1] and Mertens [3], [4] that define stable sets of equilibria and study their properties. Central to their results is the topological-structure theorem and its corollary establishing existence [1, Theorem 1 and Proposition 1]. They show that each game has a component of its Nash equilibria with the property that every perturbation of the normal form of every equivalent game has a Nash equilibrium close to this component. A component with this property is called invariant. Two games are equivalent if they have the same "reduced" normal form, in the sense that no pure strategy's payoffs are a convex combination of the payoffs from other pure strategies of the same player.

Checking directly that a component is invariant presents severe difficulties. Our current contribution is to provide a sufficient condition for invariance. We show that a component is invariant if the projection from its neighborhood in the Nash graph to the neighborhood of the game is an essential map in the homological sense. A second motive is to lay foundations for a study of stability based on payoff perturbations, which is sometimes a more powerful means of equilibrium selection than the usual approach based on strategy perturbations.[1]

**Formulation and theorem.** We consider finite normal-form games with a fixed set $N$ of players. For each player $n \in N$, let $S_n$ be the finite set of feasible pure strategies in the game, and let $\Sigma_n$ be the space of mixed strategies, represented as the simplex of probability distributions over the pure strategies. Specify $S \equiv \prod_N S_n$ and $\Sigma \equiv \prod_N \Sigma_n$. Each game $G$ is identified by its feasible strategies and the payoffs $(G_n(s))_{n \in N}$ to the players from each profile $s = (s_n)_{n \in N} \in S$ of the players' pure strategies. Thus, for each configuration of feasible strategies, interpret the space of games as $\mathscr{G} = \Re^{N \times S}$. A player $v$'s payoff from a profile $\sigma \in \Sigma$ of mixed strategies is the expected payoff: $G_v(\sigma) = \sum_S G_v(s) \prod_N \sigma_n(s_n)$.

A profile $\sigma$ of mixed strategies is a Nash equilibrium if each player $v$'s strategy is an optimal reply to the others; i.e., $G_v(\sigma) = \max_{\Sigma_v} G_v(\sigma)$, where the maximization is over the reply by $v$ to other players' strategies. Let $E$ be the graph of the correspondence $\mathscr{N}: \mathscr{G} \to \Sigma$ that maps each game to the set of its Nash equilibria;

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[1] McLennan [5] shows that a component that is essential with respect to perturbations of players' best-reply correspondences is also invariant with respect to duplication of *pure* strategies.