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# SINGULARITIES AT INFINITY AND THEIR VANISHING CYCLES

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**1. Introduction.** A polynomial $f: \mathbf{C}^n \to \mathbf{C}$ induces a locally trivial fibration $f: \mathbf{C}^n \setminus f^{-1}(\Lambda) \to \mathbf{C} \setminus \Lambda$ above the complement of a finite set $\Lambda$; this is a consequence of Thom's work [T] and a proof can be found in [V, Corollary 5.1]. The points in $\Lambda$ are either critical values of $f$ or *atypical* values coming from "the singularities at infinity of $f$."

There is up to now no definition for "singularity at infinity" of $f$. What one can only see is the effect of such a hidden thing, namely the change in the topology of the fibre (nonfiberability).

We propose a natural and rather large class of polynomials where "singularity at infinity" has a precise meaning, as follows. First we extend the function $f$ to a proper function $t: \mathbf{X} \to \mathbf{C}$. Then endow $\mathbf{X}$ with a certain Whitney stratification: $t$ becomes a stratified mapping, thus having stratified singularities. We shall call *$\mathscr{W}$-singularity at infinity* the germ at $\mathbf{X} \setminus \mathbf{C}^n$ of the singular locus of $t$.

It would be possible that some of these $\mathscr{W}$-singularities do not count as singularities at infinity of $f$ (for instance, if they do not produce atypical values). On the other hand, it appears that, like in the local case, these singularities can be nonisolated, so the hope to find more precise results than general connectivity statements would be rather small.

It was therefore reasonable for us to focus on polynomials with *isolated $\mathscr{W}$-singularities at infinity* (see also the end of Remark 5.2). In this case, we prove that "$\mathscr{W}$-singularities at infinity" is an incarnation of "singularities at infinity of $f$." This class includes the reduced plane curves and includes strictly the class of polynomials with "isolated singularities at infinity" in the sense used by Parusinski [Pa]; see our examples in 2.6.

To give an account on the difficulties one may encounter in the study of isolated $\mathscr{W}$-singularities, let us refer to the local situation. If $g: (Y, y) \to (\mathbf{C}, 0)$ is a function germ with isolated singularity with respect to a Whitney stratification of $Y$ at $y$ (see [L4] for the definition), then one may hope that $y$ being a singularity of $g$ is equivalent to existence of vanishing cycles in the Milnor fibre of $g$. This is well known to be true for smooth $Y$, but not anymore if $Y$ is singular (see [Ti] for examples).

Our main result is that the general fibre of a polynomial with isolated $\mathscr{W}$-singularity at infinity is homotopy equivalent to a wedge of spheres of dimension $n - 1$. We then relate the existence of an isolated $\mathscr{W}$-singularity at infinity to the

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