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THE MASLOV INDEX, THE SPECTRAL FLOW, AND DECOMPOSITIONS OF MANIFOLDS

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0. Introduction. Consider a closed, compact-oriented Riemann manifold (M, g) and a Clifford bundle $\mathscr{E} \to M$ over M. The spectral flow of a smooth path of selfadjoint Dirac operators D^t: $C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E})$ is the integer obtained by counting, with sign, the number of eigenvalues of D^t that cross 0 as t varies; it is a homotopy invariant of the path (cf. [AS]). The aim of this paper is to describe the spectral flow in terms of a decomposition of the manifold.

More precisely, suppose that M is divided into two manifolds-with-boundary M_1 and M_2 by an oriented hypersurface $\Sigma \subset M$. Assume that in a tubular neighborhood N of Σ, the metric is a product and the operators D^t have the "cylindrical" form

$$D^t = c(ds)(\partial/\partial s + D_0^t), \tag{0.1}$$

where s is the longitudinal coordinate in N, $c(ds)$ is the Clifford multiplication by ds, and D_0^t is independent of s. Set $\mathscr{E}_0 = \mathscr{E}|_\Sigma$ and denote by D_1^t and D_2^t the restriction of D^t to M_1 and M_2.

The kernels of D_j^t are infinite-dimensional spaces of solutions of $D_j^t\psi = 0$ on M_j. Restriction to Σ gives the *Cauchy-data spaces* (CD spaces)

$$\Lambda_1(t) = \operatorname{Ker} D_1^t|_\Sigma, \qquad \Lambda_2(t) = \operatorname{Ker} D_2^t|_\Sigma$$

in $L^2(\mathscr{E}_0)$. Note that the intersection $\Lambda_1(t) \cap \Lambda_2(t)$ is the finite-dimensional space of solutions of $D^t\psi = 0$ on M.

This setup has a rich symplectic structure. Multiplication by $c(ds)$ introduces a complex structure in $L^2(\mathscr{E}_0)$ and hence a symplectic structure in this space. The CD spaces $\Lambda_j(t)$ are then infinite-dimensional Lagrangian subspaces of $L^2(\mathscr{E}_0)$ that vary smoothly with t, and the pair $(\Lambda_1(t), \Lambda_2(t))$ is a Fredholm pair (as defined in Section 1). As in the finite-dimensional case, one can associate to a path of Fredholm pairs of Lagrangians an integer called the Maslov index. The main result of this paper is Theorem 3.14, which states that this Maslov index equals the spectral flow of the family D^t.

The Lagrangians defined by the CD spaces are infinite-dimensional, but the setup can be reduced to finite-dimensional symplectic geometry by "stretching the neck." This is done by changing the metric on M to one in which the neck is

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