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EXISTENCE AND UNIQUENESS OF MONOTONE MEASURE-PRESERVING MAPS

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Introduction. Given a pair of Borel probability measures μ and ν on $\mathbf{R}^d$, it is natural to ask whether ν can be obtained from μ by redistributing its mass in a canonical way. In the case of the line $d = 1$ the answer is clear: as long as both measures are free from atoms—$\mu[\{x\}] = \nu[\{x\}] = 0$—there is a map $y(x)$ of the line to itself for which

$$\int_{-\infty}^{x} d\mu = \int_{-\infty}^{y(x)} d\nu.$$

Uniquely determined μ-almost everywhere, this map may be taken to be nondecreasing by a suitable choice of $y(x) \in \mathbf{R} \cup \{\pm\infty\}$ at the remaining points. Interpreting μ and ν as the initial and final distribution of a one-dimensional fluid, the transformation $y(x)$ gives a rearrangement of fluid particles yielding final state ν from the initial state μ; this rearrangement is characterized by the fact that it preserves particle ordering, obviating any need for two particles to cross. Although the generalization of this construction to higher dimensions is the focus of this paper, the one-dimensional case will be pursued slightly further: when the measures are absolutely continuous with respect to Lebesgue—$d\mu(x) = f(x)dx$ and $d\nu(y) = g(y)dy$—then, formally at least (neglecting regularity issues), the fundamental theorem of calculus yields

$$y'(x)g(y(x)) = f(x). \tag{1}$$

When μ and ν measure $\mathbf{R}^d$ rather than $\mathbf{R}$, the properties of y one might hope to preserve are not clear. An answer to this question has been provided by a theorem of Brenier [1], [2], which shows under restrictions on μ and ν, that a measure-preserving transformation $y\colon (\mathbf{R}^d, \mu) \to (\mathbf{R}^d, \nu)$ can be realized as the gradient of a convex function. In particular, y will be irrotational and will not involve crossings: $(1 - t)x + ty(x) = (1 - t)x' + ty(x')$ implies $x = x'$ if $t \in [0, 1)$. Moreover, this theorem turns out to have striking applications which will shortly be indicated. Motivated by these applications, our present purpose is to extend

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