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SPHERICAL FUNCTIONS ON AFFINE LIE GROUPS

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Introduction. By spherical functions one usually means functions on the double coset space $K\backslash G/K$, where G is a group and K is a subgroup of G. This is equivalent to considering functions on the homogeneous space G/K left invariant with respect to K. More generally, if V is a fixed irreducible representation of K, for example, finite-dimensional, one can look at functions on G/K whose left shifts by elements of K span a space which is isomorphic to V as a K-module. Consideration of such functions is equivalent to consideration of functions on G/K with values in the dual representation V^* which are equivariant with respect to the left action of K. In this (and in an even more general) framework, spherical functions were studied in the works of Harish-Chandra, Helgason, and other authors [HC], [He], [W].

In the classical theory of spherical functions, G is often a real noncompact Lie group, and K is a maximal compact subgroup of G. In this case, G/K is a noncompact symmetric space. One can also consider an associated compact symmetric space G_c/K, where G_c is a compact form of G. An important class of examples is complex semisimple groups considered as real groups. In this case, $G_c = K \times K$, and K is embedded diagonally into G. The study of K-equivariant functions on G/K is then equivalent to the study of functions on K itself equivariant with respect to conjugacy. This problem makes sense for an arbitrary group K, and it turns out that equivariant functions can be explicitly constructed as traces of certain intertwining operators. In this paper we describe such functions in two cases—K is a compact simple Lie group, and K is an affine Lie group (i.e., an infinite-dimensional group whose Lie algebra is an affine Lie algebra).

The results concerning the compact group case are given in Section 1. For a compact Lie group K and a pair W, V of irreducible finite-dimensional representations of K, we consider an intertwining operator $\Phi\colon W \to W \otimes V^*$ and associate with it the function $\Psi(x) = \mathrm{Tr}|_W(\Phi x)$. This function takes values in V^* and is equivariant with respect to conjugacy, and the Peter-Weyl theorem implies that all equivariant functions can be written as linear combinations of such functions.

The next step is computation of the radial parts of the Laplace operators of K acting on conjugacy equivariant functions. This means rewriting these operators in terms of the coordinates on the set of conjugacy classes, which is the maximal

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