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THE BEST CONSTANT PROBLEM IN THE SOBOLEV EMBEDDING THEOREM FOR COMPLETE RIEMANNIAN MANIFOLDS

EMMANUEL HEBEY AND MICHEL VAUGON

I. Introduction and results. Let (M, g) be a smooth Riemannian manifold without boundary of dimension $n \geqslant 3$. It was shown recently in Hebey-Vaugon [21] that when M is compact, there exists a positive constant $C = C(M)$ such that, for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2n/(n-2)}\, dv(g)\right)^{(n-2)/n} \leqslant \frac{4}{n(n-2)\omega_n^{2/n}} \int_M |\nabla u|^2\, dv(g) + C \int_M u^2\, dv(g) \quad (S)$$

where ω_n is the volume of the standard unit sphere of $\mathbb{R}^{n+1}$. Here, $H_1^2(M)$ is the completion of $C^\infty(M)$ with respect to the standard norm

$$\|u\| = \sqrt{\int_M |\nabla u|^2\, dv(g) + \int_M u^2\, dv(g)},$$

and the embedding of $H_1^2(M)$ in $L^{2n/(n-2)}(M)$ is critical from the Sobolev viewpoint.

Note that since $4/(n(n-2)\omega_n^{2/n})$ is the best constant for which the Sobolev inequalities related to the embedding $H_1^2 \subset L^{2n/(n-2)}$ hold, (S) is an optimal inequality. Namely, let (M, g) be a smooth Riemannian manifold of dimension $n \geqslant 3$ (not necessarily compact), and let $D(M)$ be the space of smooth functions with compact support in M. It is possible to prove that, if there exist constants A and C such that for any $u \in D(M)$

$$\left(\int_M |u|^{2n/(n-2)}\, dv(g)\right)^{(n-2)/n} \leqslant A \int_M |\nabla u|^2\, dv(g) + C \int_M u^2\, dv(g),$$

then $A \geqslant S_n = 4/(n(n-2)\omega_n^{2/n})$. The argument is purely local. (See Appendix 2).

Now, a natural question is to wonder if (S) still holds on complete manifolds. This is a more delicate question than in the compact case. For instance, $H_1^2(M)$ is not any more necessarily embedded in $L^{2n/(n-2)}(M)$. Anyway, it has been proved in Hebey [19] that if the Ricci curvature of M is bounded below and if the injectivity radius of M is positive, then, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon = C(\varepsilon, M)$ such that for any $u \in H_1^2(M)$

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