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# MOUFANG TREES AND GENERALIZED HEXAGONS

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**1. Introduction.** Let $\Gamma$ be an undirected graph, let $V(\Gamma)$ denote the vertex set of $\Gamma$, and let $G$ be a subgroup of aut($\Gamma$). For $x \in V(\Gamma)$, we will denote by $\Gamma_x$ the set of vertices adjacent to $x$ in $\Gamma$. An $n$-path of $\Gamma$ for any $n \geqslant 0$ is an $(n+1)$-tuple $(x_0, x_1, \ldots, x_n)$ of vertices such that $x_i \in \Gamma_{x_{i-1}}$ for $1 \leqslant i \leqslant n$ and $x_i \neq x_{i-2}$ for $2 \leqslant i \leqslant n$. For each $x \in V(\Gamma)$ and each $i \geqslant 1$, let $G_x^{[i]}$ denote the pointwise stabilizer in $G_x$ of the set of vertices in $\Gamma$ at distance at most $i$ from $x$. We set

$$G_{x,y,\ldots,z}^{[i]} = G_x^{[i]} \cap G_y^{[i]} \cap \cdots G_z^{[i]}$$

for each subset $\{x, y, \ldots, z\}$ of $V(\Gamma)$ and each $i \geqslant 1$. The graph $\Gamma$ will be called thick if $|\Gamma_u| \geqslant 3$ for every $u \in V(\Gamma)$. An apartment of $\Gamma$ is a connected subgraph $\Delta$ such that $|\Delta_u| = 2$ for every $u \in V(\Delta)$. When there is no danger of confusion, we will often use integers to denote vertices of $\Gamma$.

A generalized $n$-gon (for $n \geqslant 2$) is a bipartite graph of diameter $n$ and girth $2n$. A generalized $n$-gon $\Gamma$ for $n \geqslant 3$ is called Moufang if $G_{1,\ldots,n-1}^{[1]}$ acts transitively on $\Gamma_n \backslash \{n-1\}$ for every $(n-1)$-path $(1, \ldots, n)$ of $\Gamma$ for some $G \leqslant$ aut($\Gamma$). In [7], Tits showed that thick Moufang $n$-gons exist only for $n = 3$, 4, 6, and 8. If $\Gamma$ is a generalized $n$-gon and $G \leqslant$ aut($\Gamma$), then $G_{0,1}^{[1]} \cap G_{0,\ldots,n} = 1$ for every $n$-path $(0, \ldots, n)$ of $\Gamma$. (This is a special case of [5, (4.1.1)]; see Theorem 2 of [9].) Thus, the following (Theorem 1 of [9]) is a generalization of Tits's result.

1.1. THEOREM. *Let $\Gamma$ be a thick connected graph, let $G \leqslant$ aut($\Gamma$), and let $n \geqslant 3$. Suppose that for each n-path $(0, 1, \ldots, n)$ of $\Gamma$,*
(i) *$G_{1,\ldots,n-1}^{[1]}$ acts transitively on $\Gamma_n \backslash \{n-1\}$, and*
(ii) *$G_{0,1}^{[1]} \cap G_{0,\ldots,n} = 1$.*
*Then $n = 3$, 4, 6 or 8.*

We will say that a graph $\Gamma$ is $(G, n)$-Moufang if it is thick and connected, and $\Gamma$, $G$, and $n$ fulfill conditions (i) and (ii) of 1.1. In this paper, we will be mainly concerned with the case that $\Gamma$ is a tree.

In [1, (3.6)], the following beautiful connection between trees and generalized polygons was established.

1.2. THEOREM. *Let $n \geqslant 3$. Suppose $\Gamma$ is a tree and $\mathscr{A}$ a family of apartments of $\Gamma$ such that*

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