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AN EXTENSION OF HÖRMANDER'S THEOREM FOR INFINITELY DEGENERATE SECOND-ORDER OPERATORS

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1. Introduction. Let $X_0, \ldots, X_n$ denote a collection of smooth vector fields defined on an open subset D of $\mathbf{R}^d$, and $c: D \to \mathbf{R}$ a smooth function. Consider the second-order differential operator

$$L := \frac{1}{2} \sum_{i=1}^{n} X_i^2 + X_0 + c. \tag{1.1}$$

Let $\mathrm{Lie}(X_0, \ldots, X_n)$ be the Lie algebra generated by the vector fields $X_0, \ldots, X_n$. According to the theorem of Hörmander [H, Theorem 1.1], L is hypoelliptic on D if the vector space $\mathrm{Lie}(X_0, \ldots, X_n)(x)$ has dimension d at every $x \in D$. Hörmander's condition characterizes hypoellipticity for operators of the form (1.1) with analytic coefficients. However, this is not the case if the vector fields $X_0, \ldots, X_n$ defining L are allowed to be smooth nonanalytic. A striking illustration of the nonnecessity of the Hörmander condition in the smooth *nonanalytic* case is provided by a result of Kusuoka and Stroock, who have made a complete study of hypoellipticity for the class of differential operators on $\mathbf{R}^3$ of the form

$$L_\sigma := \frac{\partial^2}{\partial x_1^2} + \sigma^2(x_1) \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \tag{1.2}$$

Here σ is assumed to be a C^∞ real-valued even function, nondecreasing on $[0, \infty)$, which vanishes (only) at zero. It is shown in [KS, Theorem 8.41] that L_σ is *hypoelliptic on* $\mathbf{R}^3$ *if and only if* σ *satisfies the condition* $\lim_{s \to 0+} s \log \sigma(s) = 0$. In particular, the operator L_σ corresponding to $\sigma(s) = \exp(-|s|^p)$ is hypoelliptic if p lies in the range $(-1, 0)$; however, any such operator fails to satisfy Hörmander's condition on the hyperplane $x_1 = 0$.

Let L be the operator defined in (1.1). The purpose of this paper is to establish a criterion for hypoellipticity sharper than that of Hörmander, in the case where L has smooth nonanalytic coefficients. Our main theorem (Theorem 1.0) asserts the hypoellipticity of the operator L on D under hypotheses that allow Hörmander's general condition to *fail at an exponential rate* on a collection of surfaces in D.

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