Vol. 77, No. 3 DUKE MATHEMATICAL JOURNAL © March 1995

ON WARING'S PROBLEM FOR FOUR CUBES

JÖRG BRÜDERN AND NIGEL WATT

1. Introduction. Among the famous unsettled problems in the additive theory of numbers is the conjecture that all sufficiently large natural numbers can be expressed as the sum of four cubes of natural numbers. The fundamental work of Hardy and Littlewood on Waring's problem led to the more precise formulation that the number $r(n)$ of representations of n as the sum of four positive cubes should satisfy an asymptotic formula of the shape

$$r(n) = \Gamma\left(\frac{4}{3}\right)^3 \mathfrak{S}(n)n^{1/3} + O(n^{1/3}(\log n)^{-\varrho}) \tag{1.1}$$

where ϱ is some positive constant, and $\mathfrak{S}(n)$ is the familiar singular series usually associated with four cubes. One has

$$(\log\log n)^{-c} < \mathfrak{S}(n) < (\log\log n)^{c} \tag{1.2}$$

for some $c > 0$, and all natural numbers $n \geqslant 4$, so that (1.1), if true, would confirm the original conjecture.

Hardy and Littlewood [7] have been able to establish an analogue of (1.1) for representations by sums of nine cubes. More recently, in an important paper Vaughan [14] obtained an asymptotic formula for representations by eight cubes, and at the same time showed that

$$\sum_{n \leqslant N} \left(r(n) - \Gamma\left(\frac{4}{3}\right)^3 \mathfrak{S}(n)n^{1/3}\right)^2 \ll N^{5/3}(\log N)^{-3\varrho} \tag{1.3}$$

holds for some $\varrho > 0$. In particular it follows that (1.1) holds for all but $O(N(\log N)^{-\varrho})$ positive integers n not exceeding N. Recently Boklan [6] improved Vaughan's work slightly by showing (implicitly) that any $\varrho < 1$ is admissible in (1.3).

Although we are unable to improve upon (1.3), we are able to make some progress towards the validity of (1.1) by averaging over a short interval only.

THEOREM 1. *Let $M = N^{\theta}$ where $5/6 < \theta < 1$. Then, for all but $O(M(\log N)^{-1/4})$ integers n in the range $N < n \leqslant N + M$ the asymptotic formula*

$$r(n) = \Gamma\left(\frac{4}{3}\right)^3 \mathfrak{S}(n)n^{1/3} + O(n^{1/3}(\log n)^{-1/5})$$

holds.

Received 23 February 1994. Revision received 17 June 1994.