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RIGIDITY OF THE GAUSS MAP IN COMPACT LIE GROUPS

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The *Gauss map* for a k-dimensional submanifold M in a Lie group G is defined by

$$\Phi\colon M \to G_k(\mathfrak{g})$$
$$x \mapsto (L_{x^{-1}})_* T_x M,$$

where $G_k(\mathfrak{g})$ is the Grassmann manifold consisting of all k-dimensional vector subspaces of the Lie algebra $\mathfrak{g}$ of G. In other words, for $x \in M$, $\Phi(x)$ is the left translation of the tangent space of M at x to the unit element $e \in G$. The adjoint action of G on its Lie algebra $\mathfrak{g}$ induces an action of G on $G_k(\mathfrak{g})$, which is also called the adjoint action. In this paper, we will study a class of smooth submanifolds of a compact Lie group G whose images under the Gauss map lie in an orbit of the adjoint action of G on the Grassmannian $G_k(\mathfrak{g})$. These submanifolds are particularly interesting in the study of volume-minimizing cycles in compact Lie groups. It is well known that the real cohomology classes of compact Lie groups are completely determined by the bi-invariant forms of G (which are automatically closed) (cf. [W]). By the fundamental theorem of calibrated geometry (see, for example, [HL]), the cycles which are calibrated by bi-invariant forms are volume minimizing in their real homology classes. In general, we expect the Gauss image of the submanifolds which are calibrated by bi-invariant forms to satisfy the above condition. This is true for the submanifolds calibrated by the fundamental 3-form in G due to a result of Tasaki [T]. It is also true for the submanifolds which are calibrated by the bi-invariant 5-form and the bi-invariant 10-form in $SO(6)$ due to a result in a previous paper [L]. In fact, this is our motivation in studying such submanifolds.

According to Harvey and Lawson [HL], the study of the submanifolds whose tangent bundles are contained in a given subset of the Grassmann bundle associated to the tangent bundle of the ambient manifold is called Grassmann geometry. So one can interpret this paper as the study of the Grassmann geometry in a compact Lie Group G where the given subset of the Grassmann bundle of G consists of images of a fixed vector subspace of the Lie algebra $\mathfrak{g}$ under left and right translations.

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