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A QUANTUM ANALOGUE OF THE CAPELLI IDENTITY AND AN ELEMENTARY DIFFERENTIAL CALCULUS ON $GL_q(n)$

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0. Introduction. Since the Capelli identity was discovered about a century ago [Ca1], [Ca2], [Ca3], it has played significant roles in Classical Invariant Theory (see, e.g., [My], [Wy]) and has been extended in various directions ([Tu2], [Ho], [HoU], [Na], etc.). Among them, in a recent paper, T. Hibi and the third author [HiW] found the way to a quantum analogue of the Capelli identity for $GL(2)$. The main purpose of the present paper is to generalize this result further to the quantum group $GL_q(n)$ for an arbitrary n.

The problem of generalizing the Capelli identity is, however, closely related to another basic problem in quantum group theory. The theory of quantum groups is so far based on q-analogues of universal enveloping algebras and coordinate rings on groups. What is missing here is the notion of "differentiation" with respect to the coordinates, whereas its meaning is quite obvious in the usual case. The second theme in the present paper is to fill this gap for the quantum group $GL_q(n)$. We will introduce linear operators ∂_{ij} acting on the coordinate ring so that they share some basic properties with the usual partial differentiations $\partial/\partial t_{ij}$. The Capelli identity we prove will give a strong support to regard them as a quantum analogue of the differential operators with constant coefficients.

To show our basic picture clearly, we recall first the classical situations. Let t_{ij}, $\partial/\partial t_{ij}$, and E_{ij}, E_{ij}° be the canonical coordinate functions, the corresponding partial differentiations, and the infinitesimal generators $\mathfrak{gl}(n)$ of $GL(n)$ acting on $n \times n$ matrices respectively from the right and the left. They are related in the following expression of the polarization operators:

$$E_{ij} = \sum_{\ell=1}^{n} t_{\ell i}\frac{\partial}{\partial t_{\ell j}}, \qquad E_{ij}^{\circ} = \sum_{\ell=1}^{n} t_{j\ell}\frac{\partial}{\partial t_{i\ell}}. \tag{0.1}$$

Note here that through these E_{ij}'s and E_{ij}°'s, the space of functions on $n \times n$ matrices is regarded as a left and a right $\mathfrak{gl}(n)$-module, respectively. For $GL_q(n)$ we have the quantum counterparts of E_{ij}, E_{ij}°, and t_{ij} (cf. [RTF]). It is then apparently simple to define, similarly as in [HiW], the analogous operators ∂_{ij} by solving the linear equations corresponding to (0.1). Two points are here to be looked at more carefully with this definition. First, in the classical case, we notice that the two sets of linear equations in (0.1) coincide, if we regard them as such

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