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ESTIMATES ON STOCHASTIC OSCILLATORY INTEGRALS AND ON THE HEAT KERNEL OF THE MAGNETIC SCHRÖDINGER OPERATOR

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1. Introduction. In this paper we present a method to estimate certain d-dimensional ($d \geqslant 2$) stochastic oscillatory integrals of the form

$$\mathbf{E}_{0,x}^{2t,y} \exp\left(i \int_0^{2t} A(W(s)) \circ dW(s) \right), \tag{1}$$

where $A: \mathbf{R}^d \to \mathbf{R}^d$, $W(s)$ is the d-dimensional Brownian bridge under the constraints $W(0) = x$, $W(2t) = y$ ($\mathbf{E} := \mathbf{E}_{0,x}^{2t,y}$ denotes the expectation value of the measure of W). The stochastic integral in (1) is understood as a Stratonovich integral (denoted by $\circ$). Its relation to the usual Ito integral is

$$\int_0^{2t} A(W(s)) \circ \mathrm{d}W(s) = \int_0^{2t} A(W(s))\, \mathrm{d}W(s) + \frac{1}{2} \int_0^{2t} \operatorname{div} A(W(s))\, \mathrm{d}s. \tag{2}$$

Note that the two integrals are the same for divergence-free vectorfields.

Our motivation is basically threefold. First, we came across this problem when we proved Lieb-Thirring type inequalities [LT], [LSY] on the negative eigenvalues of a three-dimensional Pauli operator with magnetic field with constant direction (see [E2]). The heart of the matter is to estimate the heat kernel $\exp(-tH_2)(x, y)$ of the two-dimensional operator $H_2 := (p - A)^2 - B$ with $B :=$ rot $A \geqslant 0$ (here $p := -i\nabla$). The Feynman-Kac formula establishes the relation between the heat kernel and the oscillatory stochastic integral ($d = 2$):

$$e^{-tH_2}(x, y) = \frac{1}{4\pi t} e^{-(x-y)^2/4t} \mathbf{E}_{0,x}^{2t,y} \exp\left(-i \int_0^{2t} A(W(s)) \circ \mathrm{d}W(s) + \frac{1}{2} \int_0^{2t} B(W(s))\, \mathrm{d}s \right) \tag{3}$$

(the conditions of its validity (see [S2], [E1]) will be all satisfied throughout this paper). Since $H_2 \geqslant 0$ (note that $H_2 = J^*J$, where $J := p_1 + ip_2 - A_1 - iA_2$), one might conjecture that e^{-tH_2} does not blow up exponentially (neither for large t, nor for large B). On the other hand, usually H_2 has plenty of ground states (see

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