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COMPARISON OF GENERALIZED THETA FUNCTIONS

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0. Introduction. The classical theta line bundle on the Jacobian $J(C)$ of a curve can be interpreted as the determinant of the full derived image of a suitable twist of the Poincaré bundle on $J(C) \times C$. Analogously, one can define theta bundles on the moduli spaces of vector bundles or, more generally, of principal bundles over C. The spaces of sections of such bundles have attracted a lot of attention in recent years, mainly because of the important role they play in conformal field theory. The dimension of the space of generalized theta functions for a semisimple and simply connected Lie group G is given by the famous Verlinde formula [BSz], [F2], [KNR]. The question we address in the present paper is to find how these dimensions change when we change the structure group of the principal bundles. By functoriality, any homomorphism of reductive groups induces a morphism between the corresponding moduli spaces and thus relates the determinant line bundles on them. The geometry of such a map was used by R. Donagi and L. Tu [DT] to produce a formula comparing the theta functions for $GL(r, \mathbb{C})$ with those for $SL(r, \mathbb{C})$. Our main result extends this comparison statement to the case of an arbitrary reductive group R and its semisimple part G. To explain it, we need to introduce some notation.

For a connected reductive algebraic group R and an element $\ell \in \pi_1(R)$, we denote by $\mathscr{M}(R, \ell)$ the moduli space of semistable principal R-bundles of topological type ℓ. Let $G = [R, R]$ be the semisimple ideal of R and Z_0 the connected component of the center of R. Set $D: R \to R/G$ to be the quotient homomorphism and $D_{\mathscr{M}}$ the induced homomorphism of moduli spaces. Fix a principal R/G bundle A of topological type $D_*\ell$ and set $\mathscr{M}(G, A) := D_{\mathscr{M}}^{-1}(A)$. (The notion of a determinant line bundle on $\mathscr{M}(R, \ell)$ is discussed in Section 2.)

THEOREM A. *Let $\mathscr{L}(s) = \lambda(s)^{\otimes -1} \in \mathrm{Pic}_{\det}(\mathscr{M}(R, \ell))$ be an ample determinant bundle. Then the following formula holds:*

$$h^0(\mathscr{M}(G, A), \mathscr{L}(s)_{|\mathscr{M}(G,A)}) \cdot h^0(\mathscr{M}(Z_0, 0), \mathscr{L}(s)_{|\mathscr{M}(Z_0,0)}) = h^1(C, K) \cdot h^0(\mathscr{M}(R, \ell), \mathscr{L}(s)),$$

where $K = G \cap Z_0$.

Since K is a finite abelian group we have $h^1(C, K) = |K|^{2g}$. Observe further that the moduli space $\mathscr{M}(Z_0, 0)$ is isomorphic to a power of the Jacobian $J^0(C)$ of C, and hence the dimension $h^0(\mathscr{M}(Z_0, 0), \mathscr{L}_{|\mathscr{M}(Z_0,0)})$ can be read from the element in the representation ring of R that determines $\mathscr{L}$. Finally, the Verlinde formula

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