INSTANTONS ON ALE SPACES, QUIVER VARIETIES, AND KAC-MOODY ALGEBRAS

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To Professor Shoshichi Kobayashi on his 60th birthday

1. Introduction.   In this paper we shall introduce a new family of varieties, which we call *quiver varieties*, and study their geometric structures. They have close relation to the singularity theory and the representation theory of the Kac-Moody algebras.

Our original motivation was to study solutions of the anti-self-dual Yang-Mills equations on a particular class of 4-dimensional noncompact complete manifolds, the so-called ALE spaces (or the ALE gravitational instantons), which were constructed by Kronheimer [Kr1]. In [KN] we gave a description of the framed moduli space of all solutions in terms of solutions of a system of quadratic equations (called the ADHM equations) for representations of a quiver on an affine, simply laced Dynkin graph. It is an analogue of the description, given by Atiyah, Drinfeld, Hitchin, and Manin [ADHM], of the moduli space for $\mathbb{R}^4$ (or S^4) in terms of solutions of a quadratic equation for certain finite-dimensional matrices.

Once we set aside their gauge-theoretic origin, there is no longer reason to restrict ourselves to affine Dynkin graphs. Definitions can be generalized to arbitrary finite graphs. We get what we call *quiver varieties*.

We study geometric structures of quiver varieties in this paper. In [Na1] it was noticed that the moduli space of anti-self-dual connections on ALE spaces has a hyper-Kähler structure, namely a Riemannian metric equipped with three endomorphisms I, J, K of the tangent bundle which satisfy the relations of quaternion algebra and are covariant constant with respect to the Levi-Civita connection:

$$I^2 = J^2 = K^2 = -1, \qquad IJ = -JI = K, \qquad \nabla I = \nabla J = \nabla K = 0.$$

The same holds for general quiver varieties. In particular, quiver varieties have holomorphic symplectic forms. We study further properties of the quiver variety, such as a natural $\mathbb{C}^*$-action, symplectic geometry, topology, and so on. As ALE spaces closely related to simple singularities, quiver varieties have very special kinds of singularities that enjoy very nice properties.