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POINTWISE ERGODIC THEOREMS FOR RADIAL AVERAGES ON SIMPLE LIE GROUPS I

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1. Introduction, definitions, and statements of results.

1.1. Measurable actions. Let us begin by recalling some well-known facts which are needed in order to establish the existence of the operators which are the subject of the present paper. Let G be a Hausdorff locally compact second countable (lcsc) group and denote by $\mathscr{B}_G$ the σ-algebra of Borel subsets of G. Let $(X, \mathscr{B}, \lambda)$ be a standard Borel space, by which we mean that $\mathscr{B}$ is a countably generated and countably separated σ-algebra, and λ a σ-finite measure on $\mathscr{B}$. G is said to have a Borel measurable action on X, if there is a map $f: G \times X \to X$, satisfying $f(g_1 g_2, x) = f(g_1, g_2 x)$, $f(e, x) = x$ for each $g \in G$ and $x \in X$, such that f is a measurable map from $(G \times X, \mathscr{B}_G \times \mathscr{B})$ to $(X, \mathscr{B})$. The G-action, which will be denoted $f(g, x) = gx$ will be called measure preserving if $\lambda(gE) = \lambda(E)$ for each $g \in G$ and $E \in \mathscr{B}$. In the sequel, by an action of G we mean a Borel-measurable measure-preserving action. However, a function on X will be called measurable if it is measurable relative to the σ-algebra obtained as the completion of $\mathscr{B}$ with respect to λ, and we do not insist that it be a Borel function. We refer to Appendix A for further discussion.

There is a natural representation of G associated with an action, by isometric automorphisms of $L^p(X)$, $1 \leqslant p \leqslant \infty$, which is given by $(\pi(g)f)(x) = f(g^{-1}x)$. As is well known, the representation π is (strongly) continuous; namely, for each $f \in L^p(X)$, $1 \leqslant p < \infty$, the map $g \mapsto \pi(g)f$ is a continuous map from G to $L^p(X)$, where we take the norm topology on $L^p(X)$. The action is called ergodic if every G-invariant set has measure zero, or its complement has measure zero. If the measure λ is finite, ergodicity is equivalent to the absence of G-invariant functions in $L^2(X)$, other that the constant functions.

To each complex bounded Borel measure μ on G, there corresponds an operator $\pi(\mu)$, with norm bounded by $\|\mu\|_1$ in every $L^p(X)$, $1 \leqslant p \leqslant \infty$, given by:

$$\pi(\mu)f(x) = \int_G f(g^{-1}x)\, d\mu(g).$$

The last equation should be interpreted as follows: Given $f \in L^p(X)$ and $f' \in L^q(X)$, where $(1/p) + (1/q) = 1$, consider the measurable function $(g, x) \mapsto f(g^{-1}x)f'(x)$

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