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ON THE HODGE STRUCTURE OF PROJECTIVE HYPERSURFACES IN TORIC VARIETIES

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The purpose of this paper is to explain one extension of the ideas of the Griffiths-Dolgachev-Steenbrink method for describing the Hodge theory of smooth (resp. quasi-smooth) hypersurfaces in complex projective spaces (resp. in weighted projective spaces). The main idea of this method is the representation of the Hodge components $H^{d-1-p,p}(X)$ in the middle cohomology group of projective hypersurfaces

$$X = \{z \in \mathbf{P}^d: f(z) = 0\}$$

in $\mathbf{P}^d = \text{Proj } \mathbf{C}[z_1, \ldots, z_{d+1}]$ using homogeneous components of the quotient of the polynomial ring $\mathbf{C}[z_1, \ldots, z_{d+1}]$ by the ideal $J(f) = \langle \partial f/\partial z_1, \ldots, \partial f/\partial z_{d+1} \rangle$. Basic references are [14], [15], [25], [30].

In this paper, we consider hypersurfaces X in compact d-dimensional toric varieties $\mathbf{P}_\Sigma$ associated with complex rational polyhedral fan Σ of simplicial cones $\mathbf{R}^d$. According to the theory of toric varieties [13], [23], [10], [24], $\mathbf{P}_\Sigma$ is defined by the gluing together of affine toric varieties $\mathbf{A}_\sigma = \text{Spec } \mathbf{C}[\check{\sigma} \cap \mathbf{Z}^d]$ $(\sigma \in \Sigma)$ where $\check{\sigma}$ denotes the dual to σ cone. Weighted projective spaces are examples of toric varieties.

M. Audin [2] first noticed that there exists another approach to the definition of the toric variety $\mathbf{P}_\Sigma$. This definition bases on the representation of $\mathbf{P}_\Sigma$ as a quotient of some Zariski open subset $U(\Sigma)$ in an affine space $\mathbf{A}^n$ by a linear diagonal action of some algebraic subgroup $\mathbf{D}(\Sigma) \subset (\mathbf{C}^*)^n$. The group of characters of $\mathbf{D}(\Sigma)$ is isomorphic to the group of classes $\text{Cl}(\Sigma)$ of divisors on $\mathbf{P}_\Sigma$ modulo the rational equivalence. The dimension n of the open set $U(\Sigma)$ equals the number of 1-dimensional cones in the fan Σ, and the dimension of $\mathbf{D}(\Sigma)$ equals $n - d$, the rank of the Picard group of $\mathbf{P}_\Sigma$. In particular, if $\mathbf{P}_\Sigma$ is smooth, then $U(\Sigma)$ is the universal torsor over $\mathbf{P}_\Sigma$ (see [22]) and $\mathbf{D}(\Sigma)$ is the torus of Neron-Severi.

The codimension of the complement

$$Z(\Sigma) = \mathbf{A}^n \setminus U(\Sigma)$$

is at least 2. So the ring of regular algebraic functions on $U(\Sigma)$ is isomorphic to

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