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ON THE ZETA FUNCTIONS OF SHIMURA VARIETIES AND PERIODS OF HILBERT MODULAR FORMS

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Introduction. In this paper, we shall study three topics that are closely tied together by the zeta functions of certain Shimura varieties. To be precise, let F be a totally real algebraic number field of degree n and let B be a quaternion algebra over F such that

$$B \otimes_{\mathbf{Q}} \mathbf{R} \cong M_2(\mathbf{R})^r \times \mathbf{H}^{n-r}, \qquad r > 0.$$

Here $\mathbf{H}$ denotes the Hamilton quaternion algebra. Fix an isomorphism of $\overline{\mathbf{Q}}$ into $\mathbf{C}$ and put $H = \mathrm{Gal}(\overline{\mathbf{Q}}/F)$. Then $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/H$ can be identified with the set of all isomorphisms of F into $\mathbf{R}$. Let Ω be the subset of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/H$ which consists of archimedean places of F at which B splits. Let $H' = \{g \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \mid g\Omega = \Omega\}$ and let F' be the fixed field of H'; the *reflex* field F' is generated over $\mathbf{Q}$ by $\sum_{\tau \in \Omega} \tau(x)$, $x \in F$. Let $G = \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$ be the algebraic group over $\mathbf{Q}$ defined by $B^\times$ and let G_A denote the adelization of G. Let $(G_A)_f$ (resp. G_∞) be the finite (resp. archimedean) part of G_A. Let $G_{\infty+}$ be the identity component of G_∞, $Z_{\infty+}$ the identity component of the center of G_∞, and K_∞ a subgroup of $G_{\infty+}$ containing $Z_{\infty+}$ such that $K_\infty/Z_{\infty+}$ is a maximal compact subgroup of $G_{\infty+}/Z_{\infty+}$. For an open compact subgroup K of $(G_A)_f$, there exists the Shimura variety S_K defined over F' whose $\mathbf{C}$-valued points $S_K(\mathbf{C})$ form an analytic space isomorphic to $G_{\mathbf{Q}} \backslash G_A/KK_\infty$. The Hasse-Weil zeta function $Z(s, S_K/F')$ of S_K over F' is determined by Langlands [L2] up to finitely many Euler factors. Apart from the terms coming from the L-indistinguishability, its main part is equal to $\prod_\pi L(s, \pi, r_1)^{m(\pi, K)}$. Here π extends over finitely many irreducible automorphic representations of G_A, $m(\pi, K) \in \mathbf{Z}$, and r_1 is a $2^r[F' : \mathbf{Q}]$-dimensional representation of the L-group ${}^LG = GL(2, \mathbf{C})^n \rtimes \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ of G.

Now it is well known (cf. Carayol [C], Taylor [T]) that, for each such π, there exists a λ-adic representation σ_λ of $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$ such that $L(s, \pi, r_0) = L(s, \sigma_\lambda)$. Here r_0 denotes the standard representation of dimension $2n$ of ${}^LG$. Since $Z(s, S_K/F')$ can be obtained from the representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/F')$ on l-adic cohomology groups of S_K, we naturally expect that there should exist some canonical way to construct a representation of H' from a representation of H. In §1, we shall first show that such construction can be performed in two ways, additive and tensorial, including the usual induction process and the dual of transfer map as special cases. We call the representation of H' so obtained in the tensorial way from σ_λ the *tensor induction* of σ_λ, which will be denoted by $\bigotimes_\Omega \mathrm{Ind}_H^{H'} \sigma_\lambda$. We shall devote the rest of §1 to a

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