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LOG ABUNDANCE THEOREM FOR THREEFOLDS

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1. Introduction. An important step in the classification of algebraic varieties is to find a good model X in a birational equivalence class. In Mori's minimal model program, we take X to be either a minimal model or a Mori fibre space.

Now the most important property of a minimal model is that K_X, the class of the canonical divisor, is nef. This means that $K_X \cdot C$ is nonnegative, for any curve C in X. (On the other hand, a Mori fibre space is always covered by rational curves.)

However, this property is purely numerical. It has been conjectured (the abundance conjecture, 6-1-14 of [10]) that much more is true, that some multiple of K_X defines a basepoint free linear system.

Moreover, an important refinement of this conjecture goes further to the category of log divisors. (Log divisors appear naturally in the classification of open varieties and in many inductive proofs. For an introduction to the minimal model program, see [4], page 28, and for the log minimal model program see the introduction to [10].) The aim of this paper is to prove this conjecture in dimension three.

1.1. THEOREM (log abundance). *Let the pair (X, Δ) consist of a threefold X and boundary Δ, such that $K_X + \Delta$ is nef and log canonical. Then $|m(K_X + \Delta)|$ is basepoint free for some m.*

With the aid of (1.1), we can establish the following birational classification of pairs (X, Δ), where X is a threefold and $K_X + \Delta$ is log terminal: The pair (X, Δ) is birational to (X', Δ') (i.e., there is a birational map such that Δ' is the strict transform of Δ) a pair with log terminal singularities, and either

(1) $|m(K_{X'} + \Delta')|$ is basepoint free for some m, or

(2) there is a morphism $\pi'\colon X' \to Y'$, which is a log Mori fibre space.

Indeed we may apply the log minimal model program to (X, Δ). Eventually either $K_X + \Delta$ is nef, in which case (1.1) applies, or (2) holds. A similar result is true if the pair (X, Δ) is log canonical.

Here are some other immediate corollaries.

COROLLARY [2], [6]. *Let (X, Δ) be a log canonical threefold. Then the log canonical ring*

$$\bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(\llcorner m(K_X + \Delta) \lrcorner))$$

is finitely generated as a $\mathbb{C}$-algebra.

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