Vol. 74, No. 3 DUKE MATHEMATICAL JOURNAL © June 1994

THE RADIAL BEHAVIOR OF A QUASICONFORMAL MAPPING

PEKKA KOSKELA

1. Introduction. Harmonic measure in simply connected domains is singular with respect to the measure μ if for each univalent analytic function f of the unit disk there is a set of full length on the unit circle whose image under f is of μ-measure zero. Øksendal [Ø1], [Ø2] verified that harmonic measure is singular with respect to the area measure. Makarov extended this result in his remarkable paper [M1] by establishing that harmonic measure is singular with respect to the λ-dimensional Hausdorff measure for any $\lambda > 1$.

A natural generalization of this result to the setting of quasiconformal mappings would be that, for a given K-quasiconformal mapping f of B^n into $\mathbb{R}^n$, there exists a set E of full $(n-1)$-dimensional area on the boundary of B^n such that the Hausdorff dimension of $f(E)$ is $n-1$. However, it is well known that this generalization fails, as is easily seen by considering the so-called snowflake map of the unit disk in the plane for which the image of each set of positive length has Hausdorff dimension strictly greater than 1; see Example 4.1 below.

We remind the reader that a K-quasiconformal mapping of B^n into $\mathbb{R}^n$ is a homeomorphism f of B^n into $\mathbb{R}^n$ that belongs to the local Sobolev space $W^{1,n}_{\mathrm{loc}}(B^n)$ and satisfies

$$|f'(x)|^n \leqslant K J_f(x)$$

for almost every $x \in B^n$. Notice that, by the analog of Beurling's theorem, f has a radial limit for each $w \in \partial B^n$ outside a set of vanishing conformal capacity; in particular, outside a set of Hausdorff dimension zero. In Theorem A below and in what follows, we denote the union of the radial limits of f for a set E on $S^{n-1} = \partial B^n$ by $f(E)$.

We establish the following result that is essentially sharp by the aforementioned example.

THEOREM A. *Let f be a K-quasiconformal mapping of B^n into $\mathbb{R}^n$. Then for each $0 < \lambda \leqslant n-1$ there exists a set E of zero λ-dimensional Hausdorff measure on S^{n-1} such that the Hausdorff dimension of the image of $S^{n-1} \setminus E$ is at most λ', where $\lambda' < n$ depends only on the dilatation K of f, λ, and n. In particular, there is a set F of*

Received 15 July 1993.

Author's research partially supported by a Rackham Faculty Fellowship and NSF grant DMS-9305742.