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REPRESENTATIONS OF AFFINE LIE ALGEBRAS, PARABOLIC DIFFERENTIAL EQUATIONS, AND LAMÉ FUNCTIONS

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Introduction. We start with consideration of the Wess-Zumino-Witten model of conformal field theory on a torus. This is, we consider an affine Lie algebra $\hat{\mathfrak{g}}$ corresponding to some simple finite-dimensional Lie algebra $\mathfrak{g}$. For technical reasons, it is more convenient to work with a twisted realization of $\hat{\mathfrak{g}}$. Next, we consider Verma modules $M_{\lambda,k}$ over $\hat{\mathfrak{g}}$. If V is a representation of the finite-dimensional algebra $\mathfrak{g}$ then by definition a vertex operator $\Phi(z)$: $M_{\lambda,k} \to M_{\nu,k} \otimes V$ is an operator-valued formal Laurent series in z satisfying the following commutation relations with the elements of $\hat{\mathfrak{g}}$:

$$\Phi(z)a \otimes t^m = ((a \otimes t^m) \otimes 1 + z^m 1 \otimes a)\Phi(z).$$

Let $M_{\lambda_i,k}$, $i = 0 \dots n$ be a collection of Verma modules such that $\lambda_0 = \lambda_n$, and let $\Phi^i(z_i)$: $M_{\lambda_i,k} \to M_{\lambda_{i-1},k} \otimes V_i$ be vertex operators. Then we can consider the following "correlation function on the torus":

$$\mathscr{F}(z_1 \dots z_n, q, h) = \mathrm{Tr}|_{M_{\lambda_0,k}}(\Phi^1(z_1) \dots \Phi^n(z_n) q^{-\partial} e^h),$$

where ∂ is the grading operator in Verma modules[1] and $h \in \mathfrak{h}_{\mathbb{R}}$. This function takes values in the module $V = V_1 \otimes \cdots \otimes V_n$, and it is the main object of our study.

Our first goal is to derive differential equations for $\mathscr{F}$. We compute $\partial\mathscr{F}/\partial z_i$ using the same technique as for the usual Knizhnik-Zamolodchikov equations (see [TK], [FR]). However, this system of equations (Theorem 3.1) is not closed: it has the form

$$z_i \frac{\partial}{\partial z_i} \mathscr{F} = A_i(z_1 \dots z_n)\mathscr{F} + \sum \pi_i(x_l) \frac{\partial}{\partial x_l} \mathscr{F},$$

where A_i are some operators in V, and the sum is taken over an orthonormal basis x_l in $\mathfrak{h}$. Since we do not have any information about $\partial\mathscr{F}/\partial x_l$, this system does not allow us to determine $\mathscr{F}$. This system of equations in another form appeared first in the paper of Bernard [Ber].

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[1] We use the symbol ∂ for the grading operator in twisted realization, reserving the standard notation d for the untwisted grading operator; see Section 1.