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ON THE CRITICAL VALUES OF L-FUNCTIONS OF $GL(2)$ AND $GL(2) \times GL(2)$

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0. Introduction. Let F be a number field with integer ring $\mathfrak{r}$. Let G be an algebraic group over $\mathbf{Q}$ such that $G(A) = GL_2(F \otimes_{\mathbf{Q}} A)$ for each $\mathbf{Q}$-algebra A. Let $\mathbf{A}$ be the adele ring of $\mathbf{Q}$ and $\mathbf{A}_f$ be its finite part. Then we define, for each open compact subgroup U of $G(\mathbf{A}_f)$, the modular variety $Y(U)$ by the quotient space $G(\mathbf{Q})\backslash G(\mathbf{A})/UZ(\mathbf{R})C_{\infty+}$, where $C_{\infty+}$ is the identity component of the standard maximal compact subgroup of $G(\mathbf{R})$, and $Z(\mathbf{R})$ is the center of $G(\mathbf{R})$. We take U to be sufficiently small so that $Y(U)$ is naturally a Riemannian manifold of dimension $2r_1 + 3r_2$ for the number of real places r_1 and complex places r_2 of F. Let ρ be an irreducible rational representation of $G(\mathbf{Q})$ into a complex vector space $V(\rho)$. When ρ is appropriately chosen, ρ induces a representation of the fundamental group of $Y(U)$, and we can define a locally constant sheaf (or a vector bundle) $\mathscr{L}(\rho)$ on $Y(U)$ whose stalk is given by $V(\rho)$. Since $\mathscr{L}(\rho)$ has a natural hermitian structure, we can speak of harmonic forms having values in $\mathscr{L}(\rho)$. On the space of cuspidal harmonic forms with values in $\mathscr{L}(\rho)$, we have the Hecke operators $T(\mathfrak{p})$ for almost all prime ideals $\mathfrak{p}$ of $\mathfrak{r}$. The space of cuspidal harmonic q-forms is non trivial only for q in the range $[r_1 + r_2, r_1 + 2r_2]$, and the eigenvalues of Hecke operators are independent of q. Thus we may assume $q = r_1 + r_2$. Writing $\mathscr{h}(U; \rho)$ for the space of cuspidal harmonic q-forms with values in $\mathscr{L}(\rho)$, we take ω in $\mathscr{h}(U; \rho)$ such that $\omega | T(\mathfrak{p}) = \lambda(T(\mathfrak{p}))\omega$ for almost all $\mathfrak{p}$. Then by [M] or [C], we can find the largest ideal N of $\mathfrak{r}$ such that there exists a common eigenform $\omega^\circ \in \mathscr{h}(U; \rho)$ invariant under $U_1(N)$ and $\omega^\circ | T(\mathfrak{p}) = \lambda(T(\mathfrak{p}))\omega^\circ$ for almost all $\mathfrak{p}$, where

$$U_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \prod_{\mathfrak{p}} GL_2(\mathfrak{r}_{\mathfrak{p}}) | a_{\mathfrak{p}}, b_{\mathfrak{p}} \in \mathfrak{r}_{\mathfrak{p}}, d_{\mathfrak{p}} - 1 \in N_{\mathfrak{p}} \text{ and } c_{\mathfrak{p}} \in N_{\mathfrak{p}}\right\}$$
$$\subset GL_2(F_{\mathbf{A}_f}).$$

For forms invariant under $U_1(N)$, we can define the Hecke operator $T(\mathfrak{n})$ for all integral ideals $\mathfrak{n}$. Because of the rationality of ρ, the system of eigenvalues $\lambda^\sigma = \{\lambda(T(\mathfrak{n}))^\sigma\}$ for $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ occurs in $\mathscr{h}(U; \rho^\sigma)$, and the field $\mathbf{Q}(\lambda)$ generated by $\lambda(T(\mathfrak{n}))$ for all $\mathfrak{n}$ is a number field (actually $\mathbf{Q}(\lambda)$ is a CM field or a totally real

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