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GEOMETRY OF BALLS IN NILPOTENT LIE GROUPS

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1. Introduction. Let G be a simply connected, real, nilpotent Lie group of dimension n. Then G can be naturally considered as $\mathbf{R}^n$ endowed with a group structure which is "polynomial" relative to the linear coordinates in $\mathbf{R}^n$. Assume that G is also given a left-invariant Riemannian structure.

The purpose of this paper is to analyze the structure of balls with respect to this Riemannian metric. Figure 1, the approximated picture of the "ball" of radius 17 in the 3-dimensional Heisenberg group (the unique Lie group whose Lie algebra is given by $[X, Y] = Z$, Z central), shows that this geometry is very rich and even exciting.

Our main result (4.2), which might seem surprising (in the light of the rather complicated figure 1) is as follows. Let $G = \gamma_1(G) \supset \cdots \supset \gamma_{c+1}(G) = 1$ be the lower central series of G. The quotients γ_i/γ_{i+1} are vector groups of dimensions, say, d_i. Then there exist coordinates in $\mathbf{R}^n$ and a constant $a > 1$ such that, for every $r > 1$, the ball of radius r (around the origin, i.e., 1_G) is contained in the (Euclidean) box with sides parallel to the coordinate axes and sizes:

$$ar, \ldots, \quad ar, (ar)^2, \ldots, (ar)^2, \quad \ldots \quad , (ar)^c, \ldots, (ar)^c$$

(where the first group in this list consists of d_1 elements, the second d_2 elements, etc.), and another box with sizes:

$$\frac{r}{a}, \ldots, \frac{r}{a}, \ \left(\frac{r}{a}\right)^2, \ldots, \left(\frac{r}{a}\right)^2, \quad \ldots \quad , \left(\frac{r}{a}\right)^c, \ldots, \left(\frac{r}{a}\right)^c$$

is contained in this ball.

This work grew from a study on the geometry of polynomial volume growth in nilpotent Lie groups. The degree of this growth is computed here as a corollary (4.11) of the main theorem. To my knowledge, this problem was attacked in the past from different directions. The analytic approach, as appears in[2], discussed the volume of V^n, where V is a neighborhood of the origin. Another approach [6], [8] used the fact that the growth of words in a cocompact lattice is equal to the growth of volume in the universal covering (but we cannot always find a cocompact lattice). Here we present a direct, geometric approach, addressing the Riemannian geometry and using explicit techniques. It should be observed that while locally, the Riemannian geometry is "Euclidean", globally (when balls with radius bigger that 1 are concerned), it is far from being Euclidean, and it is actually similar to

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