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# LIFTING MODULAR MOD $l$ REPRESENTATIONS

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**Introduction.** Fix an odd rational prime $l$ and embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$. Also identify $\overline{\mathbb{F}}_l$ with the residue field of $\mathcal{O}_{\overline{\mathbb{Q}}_l}$. By a $\lambda$-adic representation of a compact topological group $G$ we shall mean a continuous representation $\rho: G \to GL_d(\overline{\mathbb{Q}}_l)$ which factors through $GL_d(E)$ for some finite extension $E/\mathbb{Q}_l$. By a mod $l$ representation of $G$ we mean a continuous representation $\bar{\rho}: G \to GL_d(\overline{\mathbb{F}}_l)$. If $\rho$ is a $\lambda$-adic representation of $G$ then $\rho$ is conjugate to a representation with values in $GL_d(\mathcal{O}_{\overline{\mathbb{Q}}_l})$. Choosing such a conjugate representation we get (by reduction) a mod $l$ representation $\bar{\rho}: G \to GL_d(\overline{\mathbb{F}}_l)$. We will call $\bar{\rho}$ a reduction of $\rho$ and $\rho$ a lift of $\bar{\rho}$. Neither necessarily determines the other uniquely, though by the Brauer-Nesbitt theorem the semisimplification $\bar{\rho}^{ss}$ of $\bar{\rho}$ is uniquely determined by $\rho$. (Note in particular that for a character $\chi: G \to \overline{\mathbb{F}}_l^{\times}$, there is a unique reduction $\bar{\chi}$.)

If $p$ is a rational prime we shall use $G_p$ to denote a decomposition group of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at a prime above $p$ and $I_p$ to denote the inertia subgroup of $G_p$. We let $\chi_l$ denote the $l$-adic cyclotomic character, so that if $p \neq l$ is a rational prime we have $\chi_l(\mathrm{Frob}_p) = p^{-1}$.

Let $f$ be an elliptic modular cusp form of weight $k \geqslant 1$ and level $N$. If $f$ is an eigenform for the usual Hecke operators $T_p$ and $S_p$ for $p \nmid N$, then we have $f|T_p = \theta_f(T_p)f$ and $f|S_p = \theta_f(S_p)f$ where $\theta_f(T_p)$ and $\theta_f(S_p)$ are algebraic integers. Deligne has shown (following Eichler and Shimura in the case $k = 2$ and jointly with Serre in the case $k = 1$) that there is a $\lambda$-adic representation

$$\rho_f: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_l)$$

such that, if $p \nmid lN$, then $\rho_f$ is unramified at $p$ and $\rho_f(\mathrm{Frob}_p)$ has characteristic polynomial $X^2 - \theta_f(T_p)X + p\theta_f(S_p)$. Ribet has shown that $\rho_f$ is irreducible and hence uniquely determined via the above characterisation (by the Cebotarev density theorem). It is known that $\det \rho_f(c) = -1$ where $c$ is a complex conjugation. This is expressed by saying that $\rho_f$ is odd.

We shall call a $\lambda$-adic representation $\rho$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ modular if it arises from some $f$ as above. We call a mod $l$ representation $\bar{\rho}: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{F}}_l)$ modular if it is irreducible and isomorphic to the reduction of some $\rho_f$. Any such $\bar{\rho}$ is odd. Serre has conjectured that all continuous odd irreducible representations $\bar{\rho}: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{F}}_l)$ are modular. We will say $\rho$ and $\bar{\rho}$ are modular of weight $k$ and level $N$ if we can choose $f$ of weight $k$ and level $N$. We will say that $\bar{\rho}$ is modular of weight $k$

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