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WEAK TYPE-(1, 1) INEQUALITIES AND REGULARITY PROPERTIES OF ADJOINT AND NORMALIZED ADJOINT SOLUTIONS TO LINEAR NONDIVERGENCE FORM OPERATORS WITH VMO COEFFICIENTS

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1. Introduction. In [C] Luis Caffarelli developed a new approach based on the Pucci-Aleksandrov maximum principle [P] leading to $W^{2,p}$ estimates for solutions to fully nonlinear operators in the range $p > n$, where n denotes the dimension. These fully nonlinear operators are assumed to be "continuous" perturbations of operators for which $C^{1,1}$ interior estimates hold for the solutions to the associated homogeneous problems. In [E] we extended his result to the range $p > n - \varepsilon$, where ε is a possibly small number whose size depends on the parameters of ellipticity of the operator and the dimension n.

His approach also applies to obtaining the classical $W^{2,p}$ Schauder theory in the range $p \geqslant n$, for linear operators in nondivergence form with continuous coefficients. In fact, his argument gives these $W^{2,p}$ estimates when the coefficients of the corresponding operator are functions in VMO (vanishing mean oscillation). On the other hand, it is well known that, for linear elliptic operators in nondivergence form with continuous coefficients, the $W^{2,p}$ estimates hold for all $p > 1$; and recently it was shown that these estimates still hold in the same range when the coefficients are in VMO [CFL].

The classical approach to obtaining these estimates in the case of continuous coefficients is based on the following: 1. Calderón-Zygmund theory, 2. freezing the coefficients, 3. representation of the solution as a Neumann series of Calderón-Zygmund operators, and 4. interpolation inequalities. In the case of coefficients in VMO, these estimates are obtained essentially in the same way, though the freezing of the coefficients is done at every point and the commutator theorem of Calderón is used [J, p. 60] (i.e., the commutator of a Calderón-Zygmund singular integral operator and a function of bounded mean oscillation is bounded on $L^p(R^n)$, $1 < p < \infty$).

The basic tool in either of the two above results is the fact that a Calderón-Zygmund singular integral operator defines a bounded operator on $L^p(R^n)$. The proof of the latter depends on an endpoint result, the Calderón-Zygmund decomposition, and duality. In general this endpoint result, i.e., that the Riesz transforms are bounded on $L^2(R^n)$, is proved by using either Fourier transform or integration by parts.

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