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SHAPE THEORY AND ASYMPTOTIC MORPHISMS FOR C*-ALGEBRAS

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Introduction. In this paper we relate two topological invariants of separable C*-algebras. The first is the shape invariant studied by Effros and Kaminker [EK1], [EK2] and then developed further by Blackadar [B]. The second invariant is the isomorphism class of a C*-algebra in the asymptotic homotopy category $\mathscr{A}$ introduced by Connes and Higson [CH]. We prove that two separable C*-algebras are shape equivalent if and only if they determine the same class in the category $\mathscr{A}$ (see Theorem 3.9). The connection between these two invariants is established via a strong shape invariant that we introduce here. In particular we show that any homotopy invariant functor on separable C*-algebras that commutes with inductive limits automatically factors through the Connes-Higson category and is thus related to K-theory (see Theorem 3.11).

It is known that homotopy is less useful in the study of singular spaces. Many interesting C*-algebras (like those associated with infinite discrete groups, dynamical systems and inductive systems) are meant to be noncommutative substitutes for singular spaces. Therefore it is natural to look for weaker forms of homotopy which are better adapted for the study of C*-algebras, but which are still closely allied to homotopy.

In topology, this point of view led Borsuk to introduce shape theory, which has become an important area of geometric topology [MS]. Shape theory for C*-algebras is introduced by Effros and Kaminker in [EK1]. Certain key concepts like homotopy continuity and semiprojectivity are isolated there. In a related development Blackadar introduces the notion of semiprojective map and develops a shape theory for all separable C*-algebras in [B]. While formally satisfactory, the shape theory of C*-algebras has only few significant examples for which explicit computations are available [EK2], [DN].

Another generalization of homotopy theory for C*-algebras comes from the work of Connes and Higson [CH] and is based on the notion of asymptotic morphism. Roughly speaking, an asymptotic morphism from A to B is a continuous family of maps $\varphi_t\colon A \to B$ which asymptotically satisfies the axioms for $*$-homomorphisms. The asymptotic morphisms of separable C*-algebras can be composed at the level of homotopy, giving rise to a category $\mathscr{A}$ called here the asymptotic homotopy category. This category corresponds to a very flexible notion of homotopy which leads to the discovery of a concrete realization of E-theory [H], [CH] and to the

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