CORRESPONDENCES TO ABELIAN VARIETIES I

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Introduction. Let X and Y be smooth complete algebraic varieties over a fixed field κ. A correspondence from X to Y is an element of the Chow group of $X \times Y$ [F, Chapter 16]. Correspondences behave like morphisms: they push forward homology groups and pull back cohomology groups. When α is a correspondence from X to Y, we write $\alpha: X \vdash Y$. When $f: X \to Y$ is a morphism, its graph Γ_f is a subvariety of $X \times Y$, so it determines a correspondence $[\Gamma_f] \in CH(X \times Y)$.

Our question is: When is a given correspondence $\alpha \in CH(X \times Y)$ a graph of some morphism? There are some necessary conditions: (1) $\dim(\alpha) = \dim(X)$; (2) $\pi_{X*}(\alpha) = [X]$ where $\pi_X: X \times Y \to X$ is the first projection; and (3) α commutes with the diagonal, in other words, the following diagram commutes.

$$\begin{array}{ccc} X & \overset{\alpha}{\vdash} & Y \\ {\scriptstyle \Delta_X}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \Delta_Y} \\ X \times X & \underset{\alpha \times \alpha}{\vdash} & Y \times Y \end{array}$$

The conditions (1)–(3) are not sufficient in general, for example, when X is $\mathbb{P}^1$ and Y contains a rational curve (Example 2.9). It seems that the existence of rational curves complicates this kind of problem.

The goal of this paper is to show that (1)–(3) are sufficient when Y is an Abelian variety (Theorem 2.7). This is not trivial even when X is a point, Spec κ. In this case, a correspondence α is an element of the Chow group of Y, and (1) means that α is a 0-cycle, (2) means that the degree of α is 1, and (3) means that $\alpha \times \alpha$ equals $\Delta_{Y*}(\alpha)$ in the Chow group of $Y \times Y$. Our theorem says that, when (1)–(3) are satisfied, then there is a point $P \in Y$ so that $\alpha = [P]$. The difficult point is the subtlety of the condition (3). Under the conditions (1) and (2), $\alpha \times \alpha - \Delta_{Y*}(\alpha)$ is degree 0, and its Albanese class is always 0, so this condition is the equality in the Albanese kernel.

The technique of Fourier transforms of Chow groups of Abelian varieties, studied in [B] and [DM], is essential. We review their results in §1.

Notation and Convention. We work in the category of algebraic schemes over a fixed field κ. When X is a scheme, $CH(X)$ is the Chow group of X, with rational coefficients. $CH^p(X)$ is the codimension-p part of $CH(X)$.

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