Vol. 73, No. 3 DUKE MATHEMATICAL JOURNAL © March 1994

RANK ONE ELLIPTIC A-MODULES AND A-HARMONIC SERIES

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Introduction. The power series identity

$$\exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\right) = 1 - z \tag{1}$$

permits one to express the value of a Dirichlet L-series at $s = 1$ as an algebraic linear combination of logarithms of circular units. We prove in this paper an analogue of (1) for rank-one elliptic A-modules with similar significance for arithmetic over A. Our result reduces in the case $A = \mathbb{F}_q[T]$ to results of Carlitz [3], and in cases of positive genus and class number one, e.g., $A = \mathbb{F}_3[x, y]/(y^2 - x^3 + x + 1)$, to results of Thakur [22], [24]. To Thakur we owe the important idea that (1) ought to have an analogue for general A. For the proof of our result, we work out theta identities in characteristic $p > 0$ modeled on [7, Prop. 2.16, p. 29], with Riemann's definition of theta replaced by Mumford's determinantal definition. Then we open up Drinfeld's "dictionary" [5], [18] between elliptic A-modules and certain types of algebro-geometric data, look up our theta identities on the algebro-geometric side, and find that the corresponding entry on the elliptic A-module side is a relation analogous to (1) between "A-harmonic series" and the "A-exponential".

Here is how we define an analogue of the (weighted) harmonic series

$$\sum_{n=1}^{\infty}\frac{z^n}{n}$$

in global fields of positive characteristic. Let

$$A := \Gamma(X\backslash\infty, \mathcal{O}_X),$$

where $X/\mathbb{F}_q$ is a smooth projective geometrically irreducible curve of genus g, and ∞ is an $\mathbb{F}_q$-rational point. We fix an $\mathbb{F}_q$-rational meromorphic function T on X with a simple pole at ∞ and identify the completion of the fraction field of A at ∞ with the Laurent series field $\mathbb{F}_q((1/T))$. Given $0 \neq a \in \mathbb{F}_q((1/T))$ we write

$$a := \varepsilon T^{\deg a} + (\text{lower powers of } T) \qquad (0 \neq \varepsilon \in \mathbb{F}_q)$$

Received 6 May 1993. Revision received 2 September 1993.
Author partially supported by NSF grants DMS-8957375 and DMS-8902621.