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KLEINIAN GROUPS WITH SMALL LIMIT SETS

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1. Introduction. Patterson [22] and Sullivan [28], [29] discovered and explored the deep relationships between the bottom of the spectrum of the Laplacian for a hyperbolic 3-manifold and the Hausdorff dimension of the limit set of its associated Kleinian group. In particular, Sullivan proved that, if $N = \mathbf{H}^3/\Gamma$ is geometrically finite, then $\lambda_0(N) = 1$ if and only if Γ's limit set L_Γ has Hausdorff dimension $\leqslant 1$. Moreover, he proved that, if N is geometrically finite and the Hausdorff dimension $D(L_\Gamma)$ of the limit set of Γ is greater than 1, then $\lambda_0(N) = D(L_\Gamma)(2 - D(L_\Gamma))$. In this note we will investigate finitely generated Kleinian groups whose limit sets have Hausdorff dimension $\leqslant 1$ and hyperbolic 3-manifolds N (with finitely generated fundamental group) such that $\lambda_0(N) = 1$.

Let Γ be a finitely generated (nonelementary) Kleinian group with limit set L_Γ. In this note we prove that if L_Γ has Hausdorff dimension less than one, then Γ is geometrically finite and has a finite index subgroup which is quasi-conformally conjugate to a Fuchsian group of the second kind. We further prove that, if L_Γ has Hausdorff dimension 1, then it either contains a subgroup of index at most 2 which is Fuchsian (of the first kind) or it is a function group with connected domain of discontinuity. As a corollary, we prove that, if N is a topologically tame hyperbolic 3-manifold such that $\lambda_0(N) = 1$, then it is homeomorphic either to the interior of a handlebody or to an $\mathbf{R}$-bundle over a closed surface. In this note, $D(L_\Gamma)$ denotes the Hausdorff dimension of L_Γ. The main new tool is a theorem asserting that, if $\hat{\Gamma}$ is a geometrically finite subgroup of infinite index in a nonelementary Kleinian group Γ, then $D(L_{\hat{\Gamma}}) < D(L_\Gamma)$.

MAIN THEOREM. *Let Γ be a nonelementary finitely generated Kleinian group and let L_Γ denote its limit set. Then:*

1. *If $D(L_\Gamma) < 1$, then Γ is geometrically finite and Γ has a finite index subgroup which is quasiconformally conjugate to a Fuchsian group of the second kind.*
2. *If $D(L_\Gamma) = 1$, then Γ either is a function group with connected domain of discontinuity or contains a subgroup of index at most 2 which is a Fuchsian group of the first kind.*
3. *If $D(L_\Gamma) = 1$ and Γ is geometrically finite, then either Γ has a finite index subgroup which is quasiconformally conjugate to a Fuchsian group of the second kind or Γ contains a subgroup of index at most 2 which is a Fuchsian group of the first kind.*

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