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EXOTIC FOURIER TRANSFORM

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1. Introduction. Motivated by the needs of representation theory of reductive groups over a finite field, I introduced in [L1] a nonabelian Fourier transform for any finite group Γ. This is a matrix indexed by pairs consisting of an element of Γ (up to conjugacy) and of an irreducible representation of the centralizer of that element (up to isomorphism). This matrix is hermitian, unitary, and has square one; in the case where Γ is abelian, it reduces to the standard Fourier transform matrix. However the case of nonabelian Γ is also needed in representation theory; for example, the case where Γ is the symmetric group S_5 is needed for E_8 over a finite field, and certain nonabelian 2-groups are needed for the spin groups over a finite field.

In [L3] a new interpretation of these matrices was given, in terms of Γ-equivariant vector bundles over Γ. (Equivariance is with respect to the conjugation action.) There it was shown that these equivariant vector bundles form a tensor category, and that the corresponding Grothendieck ring (a commutative algebra) has its "character table" with respect to a natural basis given essentially by the entries of the nonabelian Fourier transform.

This last property is exactly the same as the one found later by Verlinde [V] for the tensor categories arising from the Wess-Zumino-Witten model in conformal field theory (see [GW]). In fact, physicists [DVVV], [DPR], have shown that the tensor category attached to Γ in [L3] has a natural place in conformal field theory.

In [L2, page xv] it is stated that the nonabelian Fourier matrices should have a generalization, obtained by allowing the Weyl group to become a noncrystallographic Coxeter group. (A heuristic theory of unipotent representations in this case is described in [L4].) In this paper we describe such a generalization in the case where the "Weyl group" is a dihedral group: this Fourier transform transforms the vector formed by the "degrees of unipotent representations" in a family to the vector formed by the "fake degrees". This is related to the Fourier transform of Wess-Zumino-Witten-Verlinde corresponding to $SL_2 \times SL_2$.

The results in [L4] suggest also the existence of a nonabelian Fourier transform arising from the Coxeter group of type H_4; this should be associated to a tensor category with 74 simple objects, which remains to be found.

Note added August 5, 1993. A Fourier transform matrix as in the previous paragraph has meanwhile been found by G. Malle (see the Appendix following this paper).

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