Vol. 73, No. 1 DUKE MATHEMATICAL JOURNAL © January 1994

BETTI NUMBERS ON A TOWER OF COVERINGS

SAI KEE YEUNG

This article grows out of an attempt to understand the behavior of Betti numbers on compact manifolds with negative sectional curvature. Models for such consideration are compact quotients of Riemannian symmetric spaces of noncompact type.

In [Ma1], [Ma2], Matsushima proves his vanishing theorems on compact quotients of Hermitian symmetric spaces. He proves that, for such a manifold, there is an integer q such that, for $i < q$, the ith Betti number β_i equals the corresponding Betti number of its compact dual. The same proof works for other symmetric spaces by the computation of Kaneyuki-Nagano [KN1], [KN2]. In particular, these results imply the vanishing of the first Betti number of compact locally symmetric spaces of rank $\geqslant 2$. Using different methods from representation theory, Borel-Wallach [BW] derives vanishing theorems for the quaternionic rank-1 spaces and the Cayley plane as well. A uniform proof for these results is obtained by Mok-Siu-Yeung [MSY] in a study of geometric superrigidity by generalizing Matsushima's argument. The only cases not covered are the real and complex hyperbolic rank-1 case. In thess cases we actually have nonvanishing results as shown by the examples of Millson [Mi].

In [DW1], [DW2], DeGeorge and Wallach obtain asymptotic behavior of Betti numbers with respect to coverings of a compact locally symmetric space of noncompact type. More precisely, let Γ be a compact lattice in a symmetric space of noncompact type G/K. Let $\{\Gamma_j\}$ be a sequence of subgroups of Γ satisfying $\Gamma_{i+1} < \Gamma_i$, $\Gamma_1 = \Gamma$ and $\bigcap_{j=1}^{\infty} \Gamma_j = \{1\}$; then the ratio $\beta_i(\Gamma_j \backslash G/K)/[\Gamma_j\colon \Gamma] \to 0$ as $j \to \infty$ for $i \neq n/2$. A natural question is how fast the growth of $\beta_i(\Gamma_j \backslash G/K)$ can be with respect to that of $[\Gamma_j\colon \Gamma]$, or, in other words, $\mathrm{vol}(\Gamma_j \backslash G/K)$. As far as the author knows, Xue [X1] and Sarnak-Xue [SX] gave the first attempts to address such a problem. Just like DeGeorge and Wallach, they considered the more general situation of multiplicities of automorphic representations. As a result, Xue [X] showed that $\beta_i(\Gamma \backslash SU(1,2)/S(U(1) \times U(2))) \leqslant c[\mathrm{vol}(\Gamma_j \backslash SU(1,2)/S(U(1) \times U(2)))]^{157/160}$, which was improved by Sarnak-Xue [SX] to $\beta_i(\Gamma_j \backslash SU(1,2)/S(U(1) \times U(2))) \ll_\varepsilon c[\mathrm{vol}(\Gamma_j \backslash SU(1,2)/S(U(1) \times U(2)))]^{(7/12)+\varepsilon}$, where Γ_j's are cocompact arithmetic lattices. Moreover, it is conjectured in [SX], among others, that, for $G = SO(n,1)$, $\beta_i(\Gamma_j \backslash G/K) \leqslant [\mathrm{vol}(\Gamma_j \backslash G/K)]^{(2i/n-1)+\varepsilon}$, $0 \leqslant i \leqslant [(n-1)/2]$, and they proved the estimate for $n = 3$. They attributed the above form of the conjecture for $SO(n,1)$ to Gromov.

Received 18 December 1992. Revision received 26 May 1993.
Author supported partially by NSF Grant No. DMS-9100383.