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LOWER BOUNDS ON THE INTERACTION BETWEEN CAVITIES CONNECTED BY A THIN TUBE

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1. Introduction. The difference between the first two eigenvalues of an elliptic operator L on a bounded domain Ω with Dirichlet boundary conditions can often be expressed in terms of geometric data for the region Ω and the effective geometry on Ω determined, for example, by an associated Agmon metric constructed from the coefficients of L. If E_1 and E_2 are the first two Dirichlet eigenvalues for Ω and L, let $\Delta E = E_2 - E_1$. We refer to ΔE as the splitting of the first two eigenvalues. It is a measure of the lack of degeneracy of the first eigenvalue. The main result of this paper is a lower bound on ΔE for the Dirichlet Laplacian on a family of generalized dumbbell regions when the diameter of the tube connecting the cavities is sufficiently small relative to the diameters of the cavities.

Let us review some cases for which ΔE has been estimated. Suppose $L \equiv -\Delta + V$ is a self-adjoint Schrödinger operator on $\Omega = \mathbf{R}^n$ with a bounded potential V. Kirsch and Simon [KS2] obtained a universal lower bound on ΔE of the type

$$\Delta E \geqslant c(R)e^{-8\sqrt{2(n\lambda R)}}, \tag{1.1}$$

where $\lambda \geqslant \sup_{x\in \mathbf{R}^n}(\sup_{E\in[E_1,E_2]}|V(x) - E|)$, $c(R)$ is a polynomial in R (without constant term), and, for any $\varepsilon > 0$, R is the radius of the smallest closed ball containing $\{x \mid V(x) > E_1 + \varepsilon^2\}$. The exponential factor in (1.1) is not believed to be optimal. It should be expressed in terms of the Agmon metric (this is the case in one-dimension; see [KS1]). On the other hand, suppose that Ω is a smooth convex bounded domain, and that V is a nonnegative smooth convex function on Ω. Singer, Wong, Yau, and Yau [SWYY] and Yu and Zhong [YZ] obtained the bounds

$$\pi^2 d^{-2} \leqslant \Delta E \leqslant (4\pi^2 n D^{-2} + 4(M - m)n^{-1}), \tag{1.2}$$

where $d \equiv \max_{x,y\in\Omega}\{|x - y|\}$ is the diameter of Ω, $D \equiv \max\{\delta \mid B(x, \delta) \subset \Omega$ for $x \in \Omega\}$, $M = \sup_{\bar{\Omega}} V$, and $m \equiv \inf_{\bar{\Omega}} V$. The origin of the exponential factor in (1.1) is the tunneling phenomenon due to the potential V. A well-known example is the case of a double-well potential. In these situations, the lower bound is determined by the

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