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METRICS WITH NONNEGATIVE ISOTROPIC CURVATURE

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The notion of curvature on isotropic two-planes on a Riemannian manifold of dimension $\geqslant 4$ was introduced in [MMr]. It is a notion that arises very naturally in the study of the second variation of area of minimal surfaces, just as sectional curvature arises in the study of geodesics. In this paper, we shall refer to "curvature on isotropic two-planes" as "isotropic curvature", whose definition we now recall.

For any real vector space V with inner product $g(\cdot, \cdot)$, let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of V, let $g(\cdot, \cdot)$ denote also the complex bilinear extension of $g(\cdot, \cdot)$ to $V_{\mathbb{C}}$, and let $\langle \cdot, \cdot \rangle$ denote the Hermitian extension of $g(\cdot, \cdot)$ to $V_{\mathbb{C}}$ which is complex linear in the first argument and conjugate linear in the second. A subspace $W \subset V_{\mathbb{C}}$ is *isotropic* if $g(w, w) = 0$ for all $w \in W$. Now let $\mathscr{R}: \Lambda^2 TM \to \Lambda^2 TM$ denote the curvature operator and also its complex linear extension to $\Lambda^2 TM \otimes_{\mathbb{R}} \mathbb{C}$. A Riemannian manifold is said to have *positive isotropic curvature* if $\langle \mathscr{R}(v \wedge w), (v \wedge w) \rangle > 0$ whenever $\mathrm{span}\{v, w\}$ is a two-dimensional isotropic subspace of $T_{\mathbb{C}}M$.

The notion of positive isotropic curvature generalizes classical curvature conditions such as strict pointwise quarter pinching and positive curvature operator. Indeed, the Morse theory of minimal two-spheres in a Riemannian manifold of positive isotropic curvature yields the following extension of the classical sphere theorem.

THEOREM [MMr]. *Let M be a compact n-dimensional Riemannian manifold without boundary, $n \geqslant 4$. If M has positive isotropic curvature, then $\pi_i(M) = \{0\}$ for $2 \leqslant i \leqslant [n/2]$ where $[x]$ denotes the integer part of $x \in \mathbb{Q}$. In particular, if M is also simply connected, then M is homeomorphic to a sphere.*

An interesting example of a nonsimply connected manifold with positive isotropic curvature is $S^1 \times S^n$, $n \geqslant 3$, with the product of the metric on S^1 and the constant curvature $+1$ metric on S^n [MMr p. 205]. Other instructive examples are the conformally flat four-manifolds with positive scalar curvature [MWo, Prop. 2.1]. These examples show that positive isotropic curvatue does *not* imply positive, or even nonnegative, Ricci tensor. It does, however, imply positive scalar curvature (Prop. 2.5 below). The examples also suggest that the fundamental group of a manifold of positive isotropic curvature may be quite large. In the first section of this paper we show that this is indeed the case by proving the following theorem.

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