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ANALYTIC HYPOELLIPTICITY, REPRESENTATIONS OF NILPOTENT GROUPS, AND A NONLINEAR EIGENVALUE PROBLEM

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1. Introduction. The aim of this article is to demonstrate that certain partial differential operators with real analytic coefficients, all of whose solutions are infinitely differentiable, admit solutions which are not real analytic. Recall the following definition.

Definition 1.1. A partial differential operator $\mathscr{L}$ is said to be analytic hypoelliptic in an open set Ω if, for any open subset $\Omega' \subset \Omega$ and any distribution $u \in \mathscr{D}'(\Omega')$ for which $\mathscr{L}u$ is real analytic in Ω', necessarily u is real analytic in Ω'.

We write C^ω to denote the class of real analytic functions in some domain. More precise regularity properties of solutions may be measured in terms of the Gevrey classes

$$G^s = \{u: \exists C < \infty \text{ such that for all } x \text{ and all } \alpha, |\partial^\alpha u(x)| \leqslant C^{1+|\alpha|}|\alpha|^{s|\alpha|}\}.$$

Then $G^1 = C^\omega$. Results will be formulated more generally, but the primary interest lies in operators which are subelliptic, so that all of their solutions are C^∞. Moreover, our operators will have multiple characteristics and nonsymplectic characteristic varieties. Although outstanding results have been obtained in both the positive and negative directions in special cases [DZ], [GS], [HH], [He], [M1], [M2], [PR], [S], [Ta1], [Ta2], [Tv], the question of precisely which operators of this type are analytic hypoelliptic remains quite open.

Our first result is as follows. Let x, y, t be coordinates in $\mathbb{R}^3$. Let $m \geqslant 2$ denote always a positive integer.

THEOREM 1.2. *Let P be a generic homogeneous polynomial, of positive degree, in two noncommuting variables. Define*

$$\mathscr{L} = P(\partial_x, \partial_y - x^{m-1}\partial_t). \tag{1.1}$$

If $m \geqslant 3$, then $\mathscr{L}$ is not analytic hypoelliptic in any neighborhood of the origin. Moreover, there exists a solution of $\mathscr{L}u = 0$ which does not belong to G^s for any $s < m$ in any neighborhood of the origin.

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