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THE DISSECTION OF RECTANGLES, CYLINDERS, TORI, AND MÖBIUS BANDS INTO SQUARES

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1. Introduction. Tutte et al. [1] consider the problem of tiling a rectangle with square tiles, all of different size. It is found that nine is the minimum number of squares and that there are exactly two rectangles that can be tiled with nine squares.

The question is posed in [1] as to whether fewer square tiles might be placed on a cylinder or torus formed from a rectangle by identifying opposite sides. Here we give a different approach to the problem than that of Tutte et al., in which the problem of a cylinder or torus is readily addressed. We recover their result, and moreover we find that neither a cylinder nor a torus can be tiled with fewer than nine different square tiles. Specifically, we find that there are exactly two rectangles that may be tiled with nine squares, that there is an additional tiling with nine squares of the cylinders formed by identifying one pair of opposite edges of each of these rectangles, and that there are no other additional tilings of cylinders or tori with nine squares.

We then extend the analysis to include Möbius bands formed from rectangles. An example of a Möbius band tiled with eight different square tiles has been given by Bracewell [2]. We find that the minimum number of square tiles for a Möbius band is two. The next smallest number of tiles is five.

In the final section we indicate how the method might be applied to the problem of perfect squares, that is, squares that can be tiled with square tiles, all of different size.

2. Formulation. We construct a matrix A which describes a tiled rectangle as follows. Number the squares in the rectangle 1 to n. Now consider a horizontal line H drawn across the rectangle along its upper edge. If this line intersects square i, then place a 1 in the ith column of the first row of A; otherwise place zero there. Now move H down the rectangle. Each time H crosses one of the horizontal edges of a square, the set of squares that H intersects will change, and we form a new row in the matrix A. For example, for the rectangle shown in Figure 1,

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

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