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INTEGER POINTS, DIOPHANTINE APPROXIMATION, AND ITERATION OF RATIONAL MAPS

JOSEPH H. SILVERMAN

Let $\phi(z) \in \mathbb{C}(z)$ be a rational map of degree at least two, say

$$\phi(z) = \frac{a_0 z^d + a_1 z^{d-1} + \cdots + a_d}{b_0 z^d + b_1 z^{d-1} + \cdots + b_d}.$$

Such a ϕ defines a holomorphic map $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, and it is a classical problem to describe the associated dynamical system, that is, to describe the points $t \in \mathbb{P}^1(\mathbb{C})$ whose orbits

$$O_\phi^+(t) = \{\phi^n(t) \colon n = 0, 1, 2, \dots\}$$

have neighborhoods satisfying certain properties. (Note that ϕ^n means the nth iterate of ϕ, not its nth power.) For basic material concerning dynamical systems on $\mathbb{P}^1$, see [1] and [3, Part 3].

Suppose now that ϕ has rational coefficients, $\phi(z) \in \mathbb{Q}(z)$. Then there are various natural arithmetic questions one can ask about the associated dynamical system. For example, if we start with a rational number $t \in \mathbb{Q}$, we can ask if its orbit contains infinitely many integers. This will certainly occur if $\phi(z) \in \mathbb{Z}[z]$ is a polynomial with integer coefficients and we take the orbit of an integer t. Similarly, it can occur for rational maps of the form $\phi(z) = a + b/(z - a)^d$, since then $\phi^2(z)$ is a polynomial. Our first result shows that these are the only possibilities.

THEOREM A. *Let $\phi(z) \in \mathbb{Q}(z)$ be a rational function of degree at least 2 and let $t \in \mathbb{Q} \cup \{\infty\} = \mathbb{P}^1(\mathbb{Q})$. If $\phi^2(z) \notin \mathbb{C}[z]$, then the orbit $O_\phi^+(t)$ contains only finitely many integer points.*

It is possible to jazz this result up in many ways, replacing $\mathbb{Q}$ by a number field, using general rings of S-integers, and most importantly, taking more than one rational map. The following result is typical, where we refer the reader to Section 1 for definitions.

THEOREM B. *Let K be a number field, let R_S be a ring of S-integers of K, and let $\phi_1, \dots, \phi_r \colon \mathbb{P}^1 \to \mathbb{P}^1$ be rational maps of degree at least two defined over K. Let Φ be the monoid of maps $\mathbb{P}^1 \to \mathbb{P}^1$ generated by the ϕ_i's under composition, and for any*

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