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ON THE FOURIER COEFFICIENTS OF HILBERT MODULAR FORMS OF HALF-INTEGRAL WEIGHT

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Introduction. To explain our main problem, let us first recall the correspondence between an elliptic modular cusp form $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ of half-integral weight $\kappa/2$ and a form of weight $\kappa - 1$, where κ is an odd integer $\geqslant 3$. Suppose that f is of level N with a multiple N of 4 and of character ψ, and that f is an eigenform of Hecke operators. Then there is a modular form $g(z) = \sum_{n=1}^{\infty} A_n e^{2\pi inz}$ of weight $\kappa - 1$ belonging to the same eigenvalues; moreover, if we normalize g so that $A_1 = 1$ and put $D(s, g) = \sum_{n=1}^{\infty} A_n n^{-s}$, then we have

$$L(s + 1 - m, \varphi) \sum_{n=1}^{\infty} a(rn^2)n^{-s} = a(r)D(s, g) \tag{1}$$

for every squarefree $r > 0$, where $L(s, \varphi)$ is the Dirichlet L-function of φ, $\varphi(n) = \psi(n)\left(\frac{-1}{n}\right)^m\left(\frac{r}{n}\right)$, and $m = (\kappa - 1)/2$. In the paper [73] in which these results were obtained, we asked the question whether there was a connection between these Fourier coefficients $a(r)$ and $D(s, q)$.

An extremely interesting and provocative answer was given by Waldspurger in [Wa]. Namely, he proved that $r^{1-\kappa/2}\overline{\psi}(r)a(r)^2/D(m, g, \overline{\varphi})$ depends only on f and $\{r\mathbf{Q}_p^{\times 2}\}_{p|N}$, where $D(s, g, \overline{\varphi}) = \sum_{n=1}^{\infty} \overline{\varphi}(n)A_n n^{-s}$. A formula of a more exact nature which gives the quotient in an explicit form was proved by Kohnen and Zagier in the case where ψ is trivial and the level of g is odd and squarefree, with some conditions on r when the level is not 1 (see [KZ] and [K]). A similar but somewhat different type of result was independently obtained by Niwa in [N] under the assumptions that $N = 4$, r is a prime, and $r \equiv (-1)^{m+1} \pmod 4$.

The purpoose of the present paper is to prove several formulas of such exact nature for the Hilbert modular forms of an arbitrary level and an arbitrary character. To describe our results, let us call F the basic number field which is totally real and denote by $\mathbf{a}$ the set of all archimedean primes of F. Then our Hilbert modular forms are functions on $H^{\mathbf{a}}$, where H is the standard upper half plane. We start with a cusp form f of half-integral weight k and (idele) character ψ, where $k = (m_v + 1/2)_{v \in \mathbf{a}}$ with $m = (m_v)_{v \in \mathbf{a}} \in \mathbf{Z}^{\mathbf{a}}$. For each fractional ideal $\mathfrak{a}$ in F the Fourier expansion of f at a cusp corresponding to $\mathfrak{a}$ has the form

$$\sum_{\xi \in F} \lambda(\xi, \mathfrak{a}; f) \exp\left(\pi i \sum_{v \in \mathbf{a}} \xi_v z_v\right)$$

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