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FLUXES, LAPLACIANS, AND KASTELEYN'S THEOREM

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1. Introduction. The genesis of this paper was an attempt to understand a problem in condensed matter physics related to questions about electron correlations, superconductivity, and electron-magnetic field interactions. The basic idea, which was proposed a few years ago, is that a magnetic field can lower the energy of electrons when the electron density is not small. Certain very specific and very interesting mathematical conjectures about eigenvalues of the Laplacian were made, and the present paper contains a proof of some of them. Furthermore, those conjectures lead to additional natural conjectures about determinants of Laplacians which we both present and prove here. It is not clear whether these determinantal theorems have physical applications but they might, conceivably in the context of quantum field theory. Some, but not all, of the results given here were announced earlier in [LE].

The setting is quantum mechanics on a graph or lattice. (All our terminology will be precisely defined in the sequel.) Physically, the vertices of our graph Λ can be thought of either as a discretization of space (i.e., replace the Laplacian by a finite difference operator), or they can be seen as locations of atoms in a solid. There are $|\Lambda|$ vertices. In the atomic interpretation the edges become electron bonds joining the atoms, and the model is known as the tight-binding model or Hückel model. The natural Laplacian $\mathscr{L}$ associated with Λ is a $|\Lambda| \times |\Lambda|$ matrix indexed by the vertices of Λ and whose diagonal elements satisfy $-\mathscr{L}_{xx} =$ number of attached edges (or valency) of vertex x. The other elements are $\mathscr{L}_{xy} = 1$ if x and y are connected by an edge, and zero otherwise.

For us it is more convenient to consider the matrix $\hat{\mathscr{L}}$ which is the Laplacian without the diagonal term, i.e., $\mathscr{L}_{xx}$ is replaced by zero. In the context of graph theory $\hat{\mathscr{L}}$ is also known as the adjacency matrix. There are three excuses for this: (i) in the solid state context, $\hat{\mathscr{L}}$ is the natural object because atoms do not bond to themselves; (ii) most of the graphs that are considered in the physics literature have constant valency, and so $\hat{\mathscr{L}}$ and $\mathscr{L}$ have the same spectrum modulo a constant which is equal to this valency; (iii) mathematically, $\hat{\mathscr{L}}$ seems to be the more natural object—from our point of view, at least—because its spectrum on a bipartite graph is always a union of pairs λ and $-\lambda$ (when $\lambda \neq 0$), as explained in Section 2. The spectrum of $\mathscr{L}$ generally does not have any such symmetry.

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