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RATIONAL AND REAL HOMOTOPY THEORY WITH ARBITRARY FUNDAMENTAL GROUPS

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1. Introduction. In "Continuous cohomology and real homotopy type" [Z], we developed a treatment of real homotopy theory for simplicial spaces analogous to the Sullivan treatment of rational homotopy theory for simplicial sets [S]. In addition, if one drops the topologies on simplicial spaces and algebras in [Z] and replaces the reals by the rationals, one recovers a somewhat-extended version of the Sullivan rational homotopy theory. In both cases, real and rational, one must assume that the simplicial spaces (sets) and algebras over $R(Q)$ are nilpotent and of finite type. In this paper we show that both treatments can be extended to arbitrary fundamental groups and actions on higher homotopy groups by working in the category of simplicial spaces (sets) over $B\pi$.

For a simplicial space (set) X with fundamental group π, we study fibrewise localization of the map $p\colon X \to B\pi$, where p induces an isomorphism on fundamental groups. The main results show that the homotopy type and homotopy classes of maps for such localizations over the reals or rationals can be captured by algebraic objects analogous to Sullivan's minimal models.

The notion of localization being considered here may also be viewed as localizing a category with respect to a set of weak equivalences. For the Quillen-Sullivan rational homotopy theory, one considers the category $\Delta\mathscr{S}_{NF}$ of nilpotent simplicial sets of finite type and, as weak equivalences, maps $f\colon X \to Y$ inducing an isomorphism on rational cohomology. For real homotopy theory one enlarges $\Delta\mathscr{S}_{NF}$ to $\Delta\mathscr{T}_{NF}$, the category of nilpotent simplicial spaces of finite type and as weak equivalences, maps which induce an isomorphism on continuous cohomology with real coefficients. In this paper we in effect consider $\Delta\mathscr{S}_{0F}$, the category of connected simplicial sets with base point and finitely generated homotopy groups and, as weak equivalences, mappings $f\colon X \to Y$ which induce isomorphisms on fundamental groups and on cohomology with local coefficients in Q vector spaces. We also consider $\Delta\mathscr{T}_{0F}$, the category of connected simplicial spaces with base point and locally Euclidean homotopy groups and, as weak equivalences, mappings $f\colon X \to Y$ which induce isomorphisms on fundamental groups and on continuous cohomology with local coefficients in R vector spaces.

It turns out in fact to be technically simpler to work with spaces having a free π action, i.e. the universal cover of spaces with fundamental group π. Basically, we redefine the main constructions of [Z], $\Delta(A)$ and $\mathscr{A}(X)$, incorporating π actions in

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