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DENSITY OF INTEGER POINTS ON AFFINE HOMOGENEOUS VARIETIES

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Section 1. Let V be an affine variety defined over $\mathbf{Z}$ by integral polynomials $f_j \in \mathbf{Z}[x_1, \ldots, x_n]$:

$$V = \{x \in \mathbf{C}^n : f_j(x) = 0, j = 1, \ldots, v\} \tag{1.1}$$

A basic problem of diophantine analysis is to investigate the asymptotics as $T \to \infty$ of

$$N(T, V) = \{m \in V(\mathbf{Z}) : \|m\| \leqslant T\} \tag{1.2}$$

where we denote by $V(A)$, for any ring A, the set of A-points of V. Hence $\|\cdot\|$ is some Euclidean norm on $\mathbf{R}^n$.

The only general method available for such problems is the Hardy-Littlewood circle method, which however has certain limitations, requiring roughly that the codimension of V in the ambient space $\mathbf{A}^n$, as well as the degree of the equations (1.1), be small relative to n. Furthermore, there are restrictions on the size of the singular sets of the related varieties:

$$V_\mu = \{x \in \mathbf{C}^n : f_j(x) = \mu_j, j = 1, \ldots, v\}, \qquad \mu = (\mu_j) \in \mathbf{C}^n.$$

We refer to [Bi] and [Sch] for a discussion of the restriction. Regardless of these restrictions, one hopes that for many more cases $N(T, V)$ can be given in the form predicted by the Hardy-Littlewood method, that is, as a product of local densities:

$$N(T, V) \sim \prod_{p < \infty} \mu_p(V) \cdot \mu_\infty(T, V), \tag{$*$}$$

where the "singular series" $\prod_{p<\infty} \mu_p(V)$ is given by p-adic densities:

$$\mu_p(V) = \lim_{k \to \infty} \frac{\# V(\mathbf{Z}/p^k\mathbf{Z})}{p^{k \dim V}}$$

and $\mu_\infty(T, V)$ is a real density—the "singular integral." Following Schmidt [Sch], we say that V is a Hardy-Littlewood system if the above asymptotics $(*)$ is valid.

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