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L^2 FLOW OF CURVE STRAIGHTENING IN THE PLANE

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0. Introduction. In recent years, a great deal of study has been done on the evolution of closed curves under flows induced by some functionals with respect to specific metrics. One of them is the "curve shortening" initiated by M. Gage [3], which is the flow along the negative L^2 gradient of the length functional relative to the L^2 metric on plane curves. As it is widely known, the study of the curve shortening problem is to understand the deformation of regular closed curves $\gamma: S^1 \to M$ moving according to the curvature normal vector $\partial\gamma/\partial t = kN = -$"the L^2 gradient of arc length". From the geometric point of view, the curve shortening evolution is clearly of significant geometric interest, since it is interesting to know the global behavior of the evolving curves while their arc lengths decrease as efficiently as possible, in some sense. It has been proved that curve shortening will make convex curves circular asymptotically [4] and also that simple closed curves with finitely many inflection points flow to the circle [6].

This paper studies the deformation of plane curves by decreasing the total squared curvature $F = \int_\gamma k^2\, ds$ of the curve, i.e., the evolution of curves along the negative gradient of F, which is referred to as "curve straightening" [8]. Depending on the representation of the curve and the metric placed on that representation, several different gradient flows can be derived, each with different geometric properties and each requiring different techniques to analyze its properties. One of the flows was extensively studied by J. Langer and D. Singer [8] and by A. Linnér [9] using functional analysis methods, particularly the Palais-Smale condition (C). This paper determines the properties of a second flow using techniques from partial differential equations.

The total squared curvature of a curve has an interesting physical interpretation: If the ends of a springy wire are linked up smoothly, then the Bernoulli-Euler theory of elastic rods tells us that the bending energy of the wire is proportional to the total squared curvature. As a matter of fact, the investigation of the total squared curvature of curves goes back to Bernoulli and Euler. One of the oldest topics in the calculus of variation is the study of elastic rods, which, according to Bernoulli's idealization, minimizes total squared curvature among curves with fixed length and satisfying given first boundary data (for more references, see [11]). In recent years there has been further work on the critical points (known as elasticae in the plane) and their properties of the total squared curvature functional for curves in various spaces [1], [7]. It is known that the closed elasticae in the plane are circles and figure eights. The curve straightening evolution is concerned with the following

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