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QUANTUM KNIZHNIK-ZAMOLODCHIKOV EQUATIONS AND HOLOMORPHIC VECTOR BUNDLES

PAVEL I. ETINGOF

Introduction. In 1984 Knizhnik and Zamolodchikov [KZ] studied the matrix elements of intertwining operators between certain representations of affine Lie algebras and found that they satisfy a holonomic system of differential equations which are now called the Knizhnik-Zamolodchikov (KZ) equations. It turned out that the KZ equations (and hence, representation theory of affine Lie algebras) are a source of a rich variety of special functions. The Gauss hypergeometric function and its various generalizations were obtained as solutions of the KZ system.

In the recent paper [FR], Frenkel and Reshetikhin considered intertwining operators between representations of quantum affine algebras. It was shown that their matrix elements satisfy a system of holonomic difference equations—a natural quantum analogue of the KZ system. It was also shown that the solutions of this difference system are very nontrivial special functions that generalize basic hypergeometric series. In particular, one of the simplest solutions is expressed in terms of the q-hypergeometric function which was introduced by Heine in the nineteenth century. This is consistent with the general idea that all reasonable special functions should come from representation theory, as matrix elements of certain representations.

The classical KZ system can be regarded as a local system—a flat structure in the trivial vector bundle on the space of sets of N distinct points on the complex plane. This fact enabled Schechtman and Varchenko to apply geometric methods to the study of the KZ equations and obtain a complete solution for a general simple Lie algebra [SV]. This interpretation can be extended to the quantum case [M; R] with the help of a specially designed discrete analogue of the notion of a local system. Other than that, the geometric meaning of the quantum KZ equations has so far remained unclear.

The goal of this paper is to introduce a new geometric interpretation of the quantum KZ equations. In Sections 1 and 2, they are linked to certain holomorphic vector bundles on a product of N equivalent elliptic curves, naturally constructed by a gluing procedure from a system of trigonometric quantum R-matrices. Meromorphic solutions of the quantum KZ equations are interpreted as sections of such a bundle. This interpretation is an analogue of the interpretation of solutions of the classical KZ equations as sections of a flat vector bundle. It yields a nontechnical proof of consistency of the quantum KZ system.

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